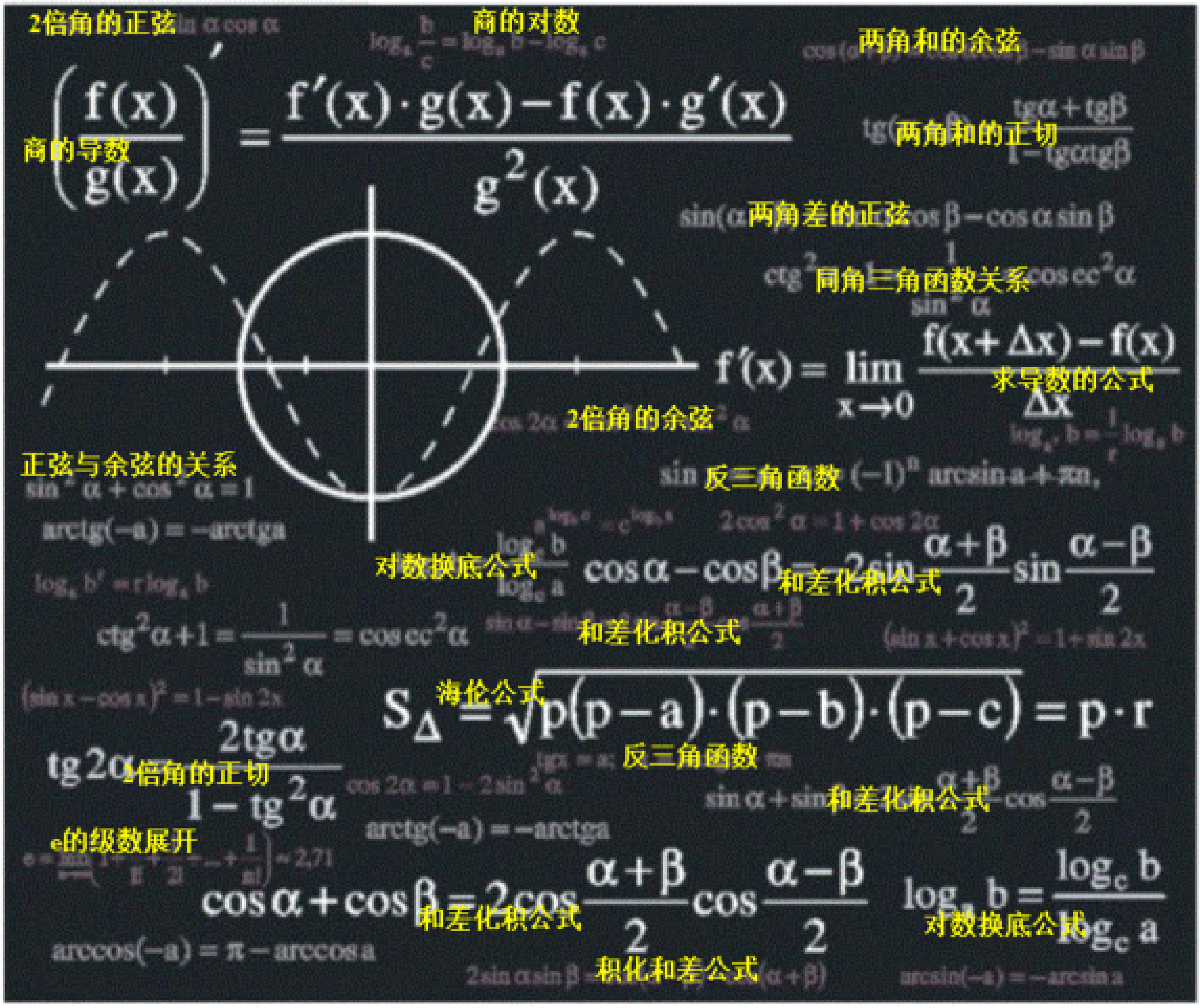

Some Methods to Calculate Limits

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1 Definition of Limits

1.1 Sequence Limit

Definition 1 a is a constant,for $\forall\varepsilon > 0$,exist $N > 0$, If $n > N$,then

$$|a_n - a| < \varepsilon$$

We call it the sequence $\{a_n\}$ converges to a

When using the definition of sequence limits to Prove limits,the main task is to find N .Usually,we solve$|a_n - a| < \varepsilon$ to find one N ,and we do not require N to be an integer.Sometimes when it is difficult to solve,we can scale the inequalities.(Because $k\varepsilon(k > 0)$ also holds)

[**e.g.1.1.1**]Prove:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Proof:Let $N = \frac{1}{\varepsilon}$,for $\forall\varepsilon > 0$, $\exists N = \frac{1}{\varepsilon}$, If $n > N$,then

$$|a_n - 0| = \frac{1}{n} < \varepsilon$$

Therefore $\lim_{n\to\infty} \frac{1}{n} = 0$ □

[**e.g.1.1.2**]Prove:

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

Proof:Let $N = \frac{1}{\varepsilon}$,for $\forall\varepsilon > 0$, $\exists N = \frac{1}{\varepsilon}$, If $n > N$,then

$$|a_n - 0| = \frac{1}{2^n} < \frac{1}{n} < \varepsilon$$

Therefore $\lim_{n\to\infty} \frac{1}{2^n} = 0$ □

[**e.g.1.1.3**]Prove:

$$\lim_{n\to\infty} \frac{n^k}{2^n} = 0(k \in N)$$

Proof:Always existing N_1 ,when $n > N_1$,then $2^n > n^{k+1}$.for $\forall\varepsilon > 0$ $\exists N = max\{N_1, \frac{1}{\varepsilon}, \}$, If $n > N$,then

$$|a_n - 0| = \frac{n^k}{2^n} < \frac{1}{n} < \varepsilon$$

Therefore $\lim_{n\to\infty} \frac{n^k}{2^n} = 0$ □

[e.g.1.1.4] If $\lim_{n\to\infty} a_n = a$ Prove:

$$\lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = a$$

Proof : Because of $\lim_{n\to\infty} a_n = a$, so $\forall\varepsilon > 0$, $\exists N > 0$, If $n > N$,then

$$|a_n - a| < \varepsilon$$

So that:

$$\begin{aligned}\left|\frac{a_1 + a_2 + \ldots + a_n}{n} - a\right| &= \left|\frac{a_1 - a + a_2 - a + \ldots + a_n - a}{n}\right| \\ &\leq \frac{|a_1 - a| + |a_2 - a| + \ldots + |a_n - a|}{n} \\ &= \frac{|a_1 - a| + |a_2 - a| + \ldots + |a_N - a|}{n} + \frac{|a_{N+1} - a| + |a_{N+2} - a| + \ldots + |a_n - a|}{n} \\ &\leq \frac{M}{n} + \frac{(n-N)\varepsilon}{n} \\ &< 2\varepsilon\end{aligned}$$

Therefore $\lim\limits_{n\to\infty} \dfrac{a_1 + a_2 + ... + a_n}{n} = a$ □

Note: After studying Stolz's Theorem, this problem will be very easy.

[e.g.1.1.5(difficult)]Let $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$,$z_n = \dfrac{x_1y_n + x_2y_{n-1} + ... + x_ny_1}{n}$Prove:

$$\lim_{n\to\infty} z_n = ab$$

Proof: Suppose$x_n = a + a_n$,$y_n = b + b_n$,so$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.So that

$$\begin{aligned}z_n &= \frac{\sum_{k=1}^{n} x_k y_{n+1-k}}{n} \\ &= \frac{\sum_{k=1}^{n} (a + a_k)(b + b_{n+1-k})}{n} \\ &= ab + \frac{\sum_{k=1}^{n} ab_{n+1-k} + \sum_{k=1}^{n} ba_k + \sum_{k=1}^{n} a_k b_{n+1-k}}{n}\end{aligned}$$

Because of $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$,so that $\forall\varepsilon > 0$, $\exists N = max\{N_1, N_2\}$, If $n > N$,then

$$|a_n| < \varepsilon, |b_n| < \varepsilon$$

When $n > N$,consider

$$\left|\frac{\sum_{k=1}^{n} ba_k}{n}\right| = \left|b\frac{a_1 + a_2 + ...a_N}{n} + b\frac{a_{N+1} + ... + a_n}{n}\right| < \frac{M}{n} + b\varepsilon < K_1\varepsilon$$

In that way

$$\left|\frac{\sum_{k=1}^{n} ab_{n+1-k}}{n}\right| < \left|\frac{M}{n} + b\varepsilon\right| < K_2\varepsilon$$

And

$$\left|\frac{\sum_{k=1}^{n} a_k b_{n+1-k}}{n}\right| \leq \frac{\sum_{k=1}^{n} |a_k b_{n+1-k}|}{n} \leq M_2 \frac{\sum_{k=1}^{n} |b_{n+1-k}|}{n} < M_3\varepsilon$$

In summary, If $n > N$,then

$$|z_n - ab| \leq \frac{|\sum_{k=1}^{n} ab_{n+1-k}| + |\sum_{k=1}^{n} ba_k| + |\sum_{k=1}^{n} a_k b_{n+1-k}|}{n} \leq (K_1 + K_2 + M_3)\varepsilon$$

Therefore $\lim\limits_{n\to\infty} z_n = ab$ □

1.2 Function Limit

Definition of Function Limit:

Definition 2 (At the limit of a certain point)Suppose$A$ is a constant,for $\forall \varepsilon > 0$,there exists $\delta > 0$, If $0 < |x - x_0| < \delta$, then

$$|f(x) - A| < \varepsilon$$

The limit of the function at point $x_0$is called A.Record as$\lim_{x \to x_0} f(x) = A$

Definition 3 (Towards the limit of positive infinity)Suppose$A$ is a constant,for $\forall \varepsilon > 0$,there exists $X > 0$, If $x > X$, then

$$|f(x) - A| < \varepsilon$$

We say that the limit of the function is A as x approaches $+\infty$.Record as$\lim_{x \to +\infty} f(x) = A$

Definition 4 (Towards the limit of negative infinity)Suppose$A$ is a constant,for $\forall \varepsilon > 0$,there exists $X > 0$, If $x < -X$, then

$$|f(x) - A| < \varepsilon$$

We say that the limit of the function is A as x approaches $-\infty$.Record as$\lim_{x \to -\infty} f(x) = A$

Definition 5 (Towards the limit of infinity)Suppose$A$ is a constant,for $\forall \varepsilon > 0$,there exists $X > 0$, If $|x| > X$, then

$$|f(x) - A| < \varepsilon$$

We say that the limit of the function is A as x approaches ∞ $\lim_{x \to \infty} f(x) = A$

Left and Right Limit:

Definition 6 (Left limit):Suppose$A$ is a constant,for $\forall \varepsilon > 0$,there exists $\delta > 0$, If $0 < x_0 - x < \delta$, then

$$|f(x) - A| < \varepsilon$$

The left limit of the function at point $x_0$is called A.Record as$\lim_{x \to x_0^-} f(x) = A$

Definition 7 (Right limit):Suppose$A$ is a constant,for $\forall \varepsilon > 0$,there exists $\delta > 0$, If $0 < x - x_0 < \delta$, then

$$|f(x) - A| < \varepsilon$$

The right limit of the function at point $x_0$is called A.Record as$\lim_{x \to x_0^+} f(x) = A$

Similar to the limit of a sequence, proving the existence of a function limit by definition also requires finding δor M.

[e.g.1.2.1]Prove:

$$\lim_{x \to 1} x = 1$$

Proof:$\forall \varepsilon > 0$,there exists $\delta = \varepsilon$, If $0 < |x - 1| < \delta$, then

$$|f(x) - 1| = |x - 1| < \varepsilon$$

So that$\lim_{x \to 1} x = 1$ □

[e.g.1.2.2]Prove:

$$\lim_{x\to x_0}\sin x=\sin x_0$$

Proof:$\forall\varepsilon>0$,there exists $\delta=\varepsilon$, If $0<|x-x_0|<\delta$, then

$$|\sin x-\sin x_0|=2\left|\cos\frac{x+x_0}{2}\right|\cdot\left|\sin\frac{x-x_0}{2}\right|\leq 2\left|\sin\frac{x-x_0}{2}\right|\leq|x-x_0|<\varepsilon$$

So that $\lim\limits_{x\to x_0}\sin x=\sin x_0$ □

2 The Four Basic Arithmetic Operations on Limits

Theorem 1 If $\lim_{n\to\infty}a_n=a$,$\lim_{n\to\infty}b_n=b$,then

$$\lim_{n\to\infty}(a_n+b_n)=a+b$$
$$\lim_{n\to\infty}(a_n-b_n)=a-b$$
$$\lim_{n\to\infty}a_n\cdot b_n=ab$$
$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{a}{b}\quad(b,b_n\neq 0)$$

Theorem 2 If $\lim_{x\to x_0}f(x)=a$,$\lim_{x\to x_0}g(x)=b$,then

$$\lim_{x\to x_0}(f(x)+g(x))=a+b$$
$$\lim_{x\to x_0}(f(x)-g(x))=a-b$$
$$\lim_{x\to x_0}f(x)\cdot g(x)=ab$$
$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{a}{b}\quad(b,g(x)\neq 0)$$

x_0 can be substituted by$x_0^+,x_0^-,+\infty,-\infty,\infty$.

[e.g.2.1]Calculate:

$$\lim_{n\to\infty}\frac{a^n}{a^n+1}\quad(a\neq-1)$$

Solution: If $a=1$,then $\lim\limits_{n\to\infty}\frac{a^n}{a^n+1}=\frac{1}{2}$

If $|a|<1$,then

$$\lim_{n\to\infty}\frac{a^n}{a^n+1}=\frac{\lim_{n\to\infty}a^n}{\lim_{n\to\infty}(a^n+1)}=0$$

If $|a|>1$,then

$$\lim_{n\to\infty}\frac{a^n}{a^n+1}=\lim_{n\to\infty}\frac{1}{1+\frac{1}{a^n}}=1$$

[**e.g.2.2**]Calculate:

$$\lim_{x\to 1}\frac{x^{m+1}-1}{x^{n+1}-1}\quad (m,n\in N)$$

Solution:Because of

$$a^n-b^n=(a-b)(a^{n-1}+a^{n-1}b+...+b^{n-1})$$

So that

$$\lim_{x\to 1}\frac{x^{m+1}-1}{x^{n+1}-1}=\lim_{x\to 1}\frac{(x-1)(x^m+x^{m-1}+...+1)}{(x-1)(x^n+x^{n-1}+...+1)}=\lim_{x\to 1}\frac{(x^m+x^{m-1}+...+1)}{(x^n+x^{n-1}+...+1)}=\frac{m+1}{n+1}\square$$

3 Squeeze Theorem(Convergence-Forcing Property)

3.1 convergence-forcing property of sequence limits

Theorem 1 for sequence a_n, b_n, c_n,there exists $N>0$, If $n>N$, then $b_n\le a_n\le c_n$,and then

$$\lim_{n\to\infty} b_n=\lim_{n\to\infty} c_n=A$$

So that the $\lim_{n\to\infty} a_n=A$

[**e.g.3.1.1**]Calculate:

$$\lim_{n\to\infty}\frac{\lfloor n\pi\rfloor}{n}$$

Solution:for all x ,there exist

$$x-1<\lfloor x\rfloor\le x$$

So that

$$n\pi-1<\lfloor n\pi\rfloor\le n\pi$$

namely

$$\pi-\frac{1}{n}<\frac{\lfloor n\pi\rfloor}{n}\le\pi$$

as well as $\lim\limits_{n\to\infty}\pi-\dfrac{1}{n}=\pi$, so that $\lim\limits_{n\to\infty}\dfrac{\lfloor n\pi\rfloor}{n}=\pi$ □

[**e.g.3.1.2**]Calculate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2+n+k}$$

Solution:Because of

$$\frac{n(n+1)}{2(n^2+2n)}=\sum_{k=1}^{n}\frac{k}{n^2+2n}\le\sum_{k=1}^{n}\frac{k}{n^2+n+k}\le\sum_{k=1}^{n}\frac{k}{n^2+n}=\frac{1}{2}$$

and

$$\lim_{n\to\infty}\frac{n(n+1)}{2(n^2+2n)}=\frac{1}{2}$$

So that $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{k}{n^2+n+k}=\dfrac{1}{2}$ □

[e.g.3.1.3(difficult)]Calculate:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^2+k}-\frac{n}{3}$$

Solution:Noticed that:

$$\sum_{k=1}^{n}\frac{k^2}{n^2}=\frac{n(n+1)(2n+1)}{6n^2}=\frac{n}{3}+\frac{1}{2}+\frac{1}{6n}$$

So that

$$\sum_{k=1}^{n}\frac{k^2}{n^2+k}-\frac{n}{3}=\sum_{k=1}^{n}\frac{k^2}{n^2+k}-\left(\sum_{k=1}^{n}\frac{k^2}{n^2}-\frac{1}{2}-\frac{1}{6n}\right)=\sum_{k=1}^{n}\left(\frac{k^2}{n^2+k}-\frac{k^2}{n^2}\right)+\frac{1}{2}+\frac{1}{6n}$$

Because of

$$\sum_{k=1}^{n}(\frac{k^2}{n^2+k}-\frac{k^2}{n^2})=-\sum_{k=1}^{n}\frac{k^3}{n^2(n^2+k)}$$

and

$$\sum_{k=1}^{n}\frac{k^3}{n^2(n^2+n)}<\sum_{k=1}^{n}\frac{k^3}{n^2(n^2+k)}<\sum_{k=1}^{n}\frac{k^3}{n^2(n^2+1)}$$

and

$$\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$$

So that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3}{n^2(n^2+k)}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3}{n^2(n^2+n)}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3}{n^2(n^2+1)}=\frac{1}{4}$$

Therefore

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k^2}{n^2+k}-\frac{k^2}{n^2}\right)+\frac{1}{2}+\frac{1}{6n}=-\frac{1}{4}+\frac{1}{2}=\boxed{\frac{1}{4}}$$

3.2 The Squeeze Theorem for Limits of Functions

Theorem 2 for function $f(x),g(x),h(x)$,there exists$\delta>0$, If $0<|x-x_0|<\delta$, then $g(x)\le f(x)\le h(x)$, and then

$$\lim_{x\to x_0}g(x)=\lim_{x\to x_0}h(x)=A$$

So that then$\lim_{x\to x_0}f(x)=A$

To use the squeeze theorem for limits of functions,The key is to find appropriate$g(x)$ and $h(x)$.

[e.g.3.2.1]Prove:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Proof: If $x>0$,Because of

$$x-\frac{x^3}{6}\le\sin x\le x$$

So that

$$1-\frac{x^2}{6}\le\frac{\sin x}{x}\le 1$$

So that

$$\lim_{x\to 0^+}\frac{\sin x}{x}=1$$

Similarly, it can be proved

$$\lim_{x\to 0^-}\frac{\sin x}{x}=1$$

So that$\lim_{x\to 0}\frac{\sin x}{x}=1$ □

4 Important Limits

4.1 $\lim_{x\to 0}\frac{\sin x}{x}=1$

Chapter4.1 The first important limit is

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

It has been proven in the previous section, so I will not elaborate further.

It's worth noting that the problems typically don't come in their original form. In fact, as long as$\square\to 0$, then $\lim_{x\to 0}\frac{\sin\square}{\square}=1$(The chapter on Equivalent Infinitesimals also has similar properties.)

[e.g.4.1.1]Calculate:

$$\lim_{x\to\pi}\frac{\sin x}{\pi-x}$$

Solution:Let$t=x-\pi$,then$\sin x=\sin t$. So that

$$\lim_{x\to\pi}\frac{\sin x}{\pi-x}=\lim_{t\to 0}\frac{\sin t}{t}=1$$

[e.g.4.1.2]Calculate:

$$\lim_{x\to 0}\frac{\tan x^2}{x^2}$$

Solution:

$$\lim_{x\to 0}\frac{\tan x^2}{x^2}=\lim_{t\to 0^+}\frac{\tan t}{t}=\lim_{t\to 0^+}\frac{\sin t}{t\cos t}=1$$

So that $\lim_{x\to 0}\frac{\tan x^2}{x^2}=1$ □

[e.g.4.1.3]Prove:

$$\lim_{x\to 0}\{\lim_{n\to\infty}[\cos x\cos\frac{x}{2}\cdots\cos\frac{x}{2^n}]\}=1$$

Proof:Because of

$$\cos x\cos\frac{x}{2}\cdots\cos\frac{x}{2^n}=\frac{\cos x\cos\frac{x}{2}\cdots\cos\frac{x}{2^n}\cdot\sin\frac{x}{2^n}}{\sin\frac{x}{2^n}}=\frac{\frac{\sin x}{2^n}}{\sin\frac{x}{2^n}}$$

So that

$$\lim_{x\to 0}\{\lim_{n\to\infty}[\cos x\cos\frac{x}{2}\cdots\cos\frac{x}{2^n}]\}=\lim_{x\to 0}\lim_{n\to\infty}\frac{\frac{\sin x}{2^n}}{\sin\frac{x}{2^n}}=\lim_{x\to 0}\frac{\sin x}{x}=1$$

4.2 $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

Here, without further proof, we introduce the second important limit:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$

This limit is equivalent to

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

[e.g.4.2.1] Calculate:

$$\lim_{x\to 0}(1+2x)^{\frac{1}{x}}$$

Solution:

$$\lim_{x\to 0}(1+2x)^{\frac{1}{x}} = \lim_{x\to 0}(1+2x)^{\frac{1}{2x}} \cdot (1+2x)^{\frac{1}{2x}} = e^2$$

[e.g.4.2.2] Calculate:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}-\frac{1}{n^2}\right)^n$$

Solution: On the one hand

$$\left(1+\frac{1}{n}-\frac{1}{n^2}\right)^n < \left(1+\frac{1}{n}\right)^n$$

On the other hand

$$\left(1+\frac{1}{n}-\frac{1}{n^2}\right)^n = (1+\frac{n-1}{n^2})^{\frac{n^2}{n-1}-\frac{n}{n-1}} \geq (1+\frac{n-1}{n^2})^{\frac{n^2}{n-1}-2}$$

as well as

$$\lim_{n\to\infty}\left(1+\frac{n-1}{n^2}\right)^{\frac{n^2}{n-1}-2} = \lim_{n\to\infty}\left(1+\frac{n-1}{n^2}\right)^{\frac{n^2}{n-1}} = e = \lim_{n\to\infty}(1+\frac{1}{n})^n$$

So that $\lim_{n\to\infty}\left(1+\frac{1}{n}-\frac{1}{n^2}\right)^n = e$ $\square$

4.3 (Supplement) $\lim_{x\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)=\gamma$

sequence $a_n = \sum_{k=1}^{n}\frac{1}{k}-\ln n$ is convergent, and its limit is known as Euler's constant. Record as γ. Below is the proof of its convergence.

[e.g.4.3.1] Proof above sequence convergence:

Proof:Because of

$$\frac{1}{n+1} < \ln(1+\frac{1}{n}) < \frac{1}{n}$$

So that

$$\sum_{k=1}^{n-1}\ln(1+\frac{1}{k}) < \sum_{k=1}^{n-1}\frac{1}{n}$$

namely

$$\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{n} > \ln n$$

So that

$$a_n > \frac{1}{n} > 0$$

And because

$$a_{n+1}-a_n = \ln\left(1-\frac{1}{n+1}\right)+\frac{1}{n+1} < 0$$

So that a_n monotonically decreasing with a lower bound , So taht a_n convergence. $\square$

[e.g.4.3.2]Calculate:

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}\right)$$

Solution:

$$a_n=\sum_{k=1}^{n}\frac{1}{k}-\ln n$$

So that

$$\begin{aligned}&\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}\right]\\&=\lim_{n\to\infty}\left[\left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{2n}-\ln 2n)-(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}-\ln n\right)\right]+\ln 2\\&=\lim_{n\to\infty}(a_{2n}-a_n+\ln 2)\\&=\ln 2\end{aligned}$$

So that $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}\right)=\ln 2$ □

[e.g.4.3.3]Calculate:

$$\lim_{n\to\infty}\left(\frac{1}{kn+1}+\frac{1}{kn+2}+...+\frac{1}{mn}\right)\quad(m,k\in N^+)$$

Hint: By following the same method as the previous problem, you can immediately obtain the limit is $\ln\frac{m}{k}$ □

5 Cauchy Convergence Criterion

Theorem 1 a_n is a sequence , If $\forall\varepsilon>0$,there exists $N>0$, If $m,n>N$, then

$$|a_m-a_n|<\varepsilon$$

We called sequence $\{a_n\}$ convergence.

corollary 1 a_n is a sequence , If $\forall\varepsilon>0$,there exists $N>0$, If $n>N$, for $\forall p\in N$ then

$$|a_{n+p}-a_n|<\varepsilon$$

We called sequence $\{a_n\}$ convergence.

[e.g.5.1]Proof sequence a_n convergence:

$$a_n=\frac{1}{1^2}+\frac{1}{2^2}+...+\frac{1}{n^2}$$

Proof:$\forall\varepsilon > 0$,there exists $N = \frac{1}{\varepsilon}$, If $n > N$, for $\forall p \in N$ then

$$
\begin{aligned}
&|a_{n+p} - a_n| \\
&= \left|\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + ... + \frac{1}{(n+p)^2}\right| \\
&\leq \left|\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + ... + \frac{1}{(n+p-1)(n+p)}\right| \\
&= \left|\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + ... + \frac{1}{n+p-1} - \frac{1}{n+p}\right| \\
&= \left|\frac{1}{n} - \frac{1}{n+p}\right| \\
&\leq \frac{1}{n} \\
&< \varepsilon
\end{aligned}
$$

Q.E.D. □

[e.g.5.2]Proof of the Divergence of the Partial Sum Sequence ($S_n = \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n}$) of the Harmonic Series ($\sum_{n=1}^{\infty} \frac{1}{n}$) sequence :

Proof:$\exists\varepsilon = \frac{1}{2}$, for $\forall N > 0$,namely $n > N$,there exists$p = n$,still

$$|S_{n+p} - S_n| = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+p} \geq \frac{p}{n+p} = \frac{1}{2} = \varepsilon$$

So that sequence S_n divergence. □

6 Equivalent Infinitesimals

Definition 1 (Infinitesimal): Suppose function $f(x)$, if $\lim_{x\to x_0} f(x) = 0$,then we called $f(x)$ is an infinitesimal as $x \to x_0$.x_0 can be substituted by$x_0^+, x_0^-, +\infty, -\infty, \infty$.

Definition 2 Equivalent Infinitesimals : Suppose function $f(x), g(x)$, if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$,then we called $f(x), g(x)$ are equivalent infinitesimals as $x \to x_0$.

When calculating limits, equivalent infinitesimals in the overall expression can be substituted. Common equivalent infinitesimals equivalent infinitesimals:

$$x \sim \sin x \sim \tan x \sim \arctan x \sim \arcsin x \sim e^x - 1 \sim \ln(1+x)$$

$$1 - \cos x \sim \frac{1}{2}x^2$$

$$(1+x)^\alpha - 1 \sim \alpha x$$

$$a^x - 1 \sim x \ln a$$

All of the above are equivalent infinitesimals as $x \to 0$.

Note: Similar to Important Limits, if $\square \to 0$,then$\square \sim \sin\square$.The same applies to the remaining functions.

[e.g.6.1]Calculate:

$$\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{1-\cos x}$$

Solution:

$$\begin{aligned}&\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{1-\cos x}\\&=\lim_{x\to 0}\frac{\frac{1}{2}x^2}{\frac{1}{2}x^2}\\&=\boxed{1}\end{aligned}$$

[e.g.6.2]Calculate:

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}\ (a,b\geq 0)$$

Solution:

(1) If at least one of a and b is 0,then the original limit is 0;

(2) If neither a nor b is 0,then:

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}=e^{\lim_{x\to 0}\frac{1}{x}\cdot\ln\left(\frac{a^x+b^x}{2}\right)}$$

Consider

$$\begin{aligned}&\lim_{x\to 0}\frac{1}{x}\cdot\ln\left(\frac{a^x+b^x}{2}\right)\\&=\lim_{x\to 0}\frac{1}{x}\cdot\ln\left(\frac{a^x+b^x}{2}-1+1\right)\\&=\lim_{x\to 0}\frac{1}{x}\cdot\left(\frac{a^x-1+b^x-1}{2}\right)\\&=\lim_{x\to 0}\frac{1}{x}\cdot\left(\frac{a^x-1}{2}\right)+\lim_{x\to 0}\frac{1}{x}\cdot\left(\frac{b^x-1}{2}\right)\\&=\lim_{x\to 0}\frac{1}{x}\cdot\left(\frac{x\ln a}{2}\right)+\lim_{x\to 0}\frac{1}{x}\cdot\left(\frac{x\ln b}{2}\right)\\&=\ln\sqrt{ab}\end{aligned}$$

So that $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}=\sqrt{ab}$ □

Note: And so on, or similarly by inference,we get

$$\lim_{x\to 0}\left(\frac{a_1^x+a_2^x+...+a_n^x}{n}\right)^{\frac{1}{x}}=\sqrt[n]{a_1a_2\cdots a_n}\ (a_i\geq 0, i=1,2...n)$$

[e.g.6.3]Calculate:

$$\lim_{x\to\infty}\frac{\sin x}{x}$$

Solution:Note that the limit is tending towards ∞,so $\lim_{x\to\infty}\frac{\sin x}{x}=0$ □

[e.g.6.4]Calculate:

$$\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}$$

Solution:

$$\begin{aligned}&\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}\\&\lim_{x\to 0}\frac{2\sin x-2\sin x\cos x}{x^3}\\&\lim_{x\to 0}\frac{2\sin x(1-\cos x)}{x^3}\\&\lim_{x\to 0}\frac{2x(\frac{1}{2}x^2)}{x^3}\\&=\boxed{1}\end{aligned}$$

Note: This problem cannot be solved directly by substituting $2\sin x\sim 2x$ and $\sin 2x\sim 2x$. Because when substituting equivalent infinitesimals, the substitution needs to be done as a whole. Replacing individual terms within a factor may lead to errors.

7 Heine's Theorem

Theorem 1 Suppose $f(x)$ is defined on $U^o(x_0,\delta')$, then a necessary and sufficient condition for the existence of $\lim_{x\to x_0} f(x)$ is that for any sequence $\{x_n\}$ contained in $U^o(x_0,\delta')$ and converging to x_0 the limit $\lim_{n\to\infty} f(x_n)$ exists and is the same for all such sequences.

The Heine's Theorem is generally used to determine the non-existence of a limit or to convert problems of finding limits of sequences into problems of finding limits of functions. If there exists a sequence $\{x_n\}$, such that the limit $\lim_{n\to\infty} f(x_n)$ does not exist, or if there exist two sequences $\{x_n\},\{y_n\}$, such that the limits $\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} f(y_n)$ exist but are not equal, then the original function limit does not exist.

[e.g.7.1]Prove that the following limit does not exist.

$$\lim_{x\to+\infty}\sin(\frac{\pi x}{2})$$

Proof: We can take two subsequences $x_n=2n$ and $y_n=4n+1$, then we have:

$$\lim_{n\to\infty}\sin(\frac{2n\pi}{2})=0$$
$$\lim_{n\to\infty}\sin\left[\frac{(4n+1)\pi}{2}\right]=1$$

Therefore, the limit of the original function does not exist. □

[e.g.7.2]Calculate:

$$\lim_{n\to\infty} n^2\left(1-\cos\frac{1}{n}\right)$$

Solution: We only need to calculate

$$\lim_{x\to+\infty} x^2\left(1-\cos\frac{1}{x}\right)$$

Because of

$$\lim_{x\to+\infty} x^2\left(1-\cos\frac{1}{x}\right)=\lim_{x\to+\infty} x^2\cdot\frac{1}{2x^2}=\frac{1}{2}$$

Therefore, according to Heine's Theorem, we know that

$$\lim_{n\to\infty} n^2\left(1-\cos\frac{1}{n}\right)=\boxed{\frac{1}{2}}$$

8 L'Hospital's Rule

8.1 Indeterminate Form of $\frac{0}{0}$

Theorem 1 If $f(x)$ and $g(x)$ satisfy:
(1)$\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x)=0$;
(2)$f(x)$ and $g(x)$ are differentiable on the punctured neighborhood $U^o(x_0,\delta)$ of x_0 , and $g'(x)\neq 0$;
(3)$\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=A$ Then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=A$$

Note :x_0 can be replaced by $x_0^+, x_0^-, +\infty, -\infty, \infty$, and A can be $+\infty, -\infty, \infty$.

[e.g.8.1.1]Calculate:

$$\lim_{x\to0^+}\frac{\sqrt{x}}{1-e^{\sqrt{x}}}$$

Solution:Let $t=\sqrt{x}$,Then

$$\begin{aligned}&\lim_{x\to0^+}\frac{\sqrt{x}}{1-e^{\sqrt{x}}}\\&=\lim_{t\to0^+}\frac{t}{1-e^t}\\&=\lim_{t\to0^+}\frac{1}{-e^t}\\&=\boxed{-1}\end{aligned}$$

[e.g.8.1.2]Calculate:

$$\lim_{x\to 0}\frac{e^x-\sqrt{1+2x}}{\ln(1+x^2)}$$

Solution:

$$\begin{aligned}
&\lim_{x\to 0}\frac{e^x-\sqrt{1+2x}}{\ln(1+x^2)}\\
&=\lim_{x\to 0}\frac{e^x-\sqrt{1+2x}}{x^2}\\
&=\lim_{x\to 0}\frac{e^x-(1+2x)^{-\frac{1}{2}}}{(2x)}\\
&=\lim_{x\to 0}\frac{e^x+(1+2x)^{-\frac{3}{2}}}{(2)}\\
&=\boxed{1}
\end{aligned}$$

[e.g.8.1.3]Calculate:

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}$$

Solution:

$$\begin{aligned}
&\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}\\
&=\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e}{x}\\
&=\lim_{x\to 0}e\frac{e^{\frac{\ln(1+x)}{x}-1}-1}{x}\\
&=\lim_{x\to 0}e\frac{\frac{\ln(1+x)}{x}-1}{x}\\
&=\lim_{x\to 0}e\frac{\ln(1+x)-x}{x^2}\\
&=\lim_{x\to 0}e\frac{\frac{1}{1+x}-1}{2x}\\
&=\lim_{x\to 0}-e\frac{1}{2(1+x)}\\
&=\boxed{-\frac{e}{2}}
\end{aligned}$$

8.2 Indeterminate Form of $\frac{*}{\infty}$

Theorem 2 If $f(x)$ and $g(x)$ satisfy:
(1)$\lim_{x\to x_0} g(x)=\infty$;
(2)$f(x)$ and $g(x)$ are differentiable on the punctured neighborhood $U^o(x_0,\delta)$ of x_0, and$g'(x)\neq 0$;
(3)$\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=A$ Then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=A$$

Note :x_0 can be replaced by $x_0^+, x_0^-, +\infty, -\infty, \infty$,$A$ can be $+\infty, -\infty, \infty$.

[e.g.8.2.1]Calculate:

$$\lim_{x\to+\infty}\frac{x^{2024}}{e^x}$$

Solution:

$$\begin{aligned}&\lim_{x\to+\infty}\frac{x^{2024}}{e^x}\\&=\lim_{x\to+\infty}\left(\frac{x}{e^{\frac{x}{2024}}}\right)^{2024}\\&=\left(\lim_{x\to+\infty}\frac{x}{e^{\frac{x}{2024}}}\right)^{2024}\\&=\left(\lim_{x\to+\infty}\frac{2024}{e^{\frac{x}{2024}}}\right)^{2024}\\&=\boxed{0}\end{aligned}$$

[e.g.8.2.2]Calculate:

$$\lim_{x\to+\infty}\frac{x+\sin x}{x}$$

Solution:

$$\begin{aligned}&\lim_{x\to+\infty}\frac{x+\sin x}{x}\\&=\lim_{x\to+\infty}1+\frac{\sin x}{x}\\&=1+\lim_{x\to+\infty}\frac{\sin x}{x}\\&=\boxed{1}\end{aligned}$$

[e.g.8.2.3(Fallible)]The function $f(x)$ is second-order differentiable at $x=0$, and $\lim_{x\to0}\frac{f(x)}{x^3}=1$.Then, which of the following statements are correct?

$$A.\lim_{x\to0}\frac{f'(x)}{x^2}=3\quad B.f''(0)=0$$
$$C.\lim_{x\to0}\frac{f''(x)}{x}=3\quad D.f'''(0)=0$$

Solution:Let

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{6}x^3+o(x^3)$$

Then

$$\begin{aligned}&\lim_{x\to0}\frac{f(x)}{x^3}=1\\&=\lim_{x\to0}\frac{f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{6}x^3+o(x^3)}{x^3}=1\end{aligned}$$

Therefore, for the limit to exist, it is necessary that $f(0)=f'(0)=f''(0)=0$,$f'''(0)=6$

From this, we can conclude that option B is correct and option D is incorrect.

For options A and C, we can provide counterexamples: $f(x)=\begin{cases}0, & \text{if } x=0\\ x^3+x^4\cdot\sin\frac{1}{x}, & \text{if } x\neq0\end{cases}$

9 Taylor Formula

9.1 Taylor's Formula with Peano's Remainder

Definition 1 If $f(x)$ has nth-order derivatives at $x = x_0$,then $f(x)$ can be expressed in the following form:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o(x - x_0)^n$$

When $x_0 = 0$,it is called Maclaurin's formula. Some commonly used Maclaurin's formulas include:

$$\begin{aligned}
&e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + o(x^n)\\
&\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + .. + (-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!} + o(x^{2m})\\
&\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + .. + (-1)^m\frac{x^{2m}}{(2m)!} + o(x^{2m+1})\\
&\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + (-1)^{n-1}\frac{x^n}{n} + o(x^n)\\
&(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2!}x^2 + ... + \frac{\alpha(\alpha - 1)\cdots(\alpha - n + 1)}{n!}x^n + o(x^n)\\
&\frac{1}{1 - x} = 1 + x + x^2 + ... + x^n + o(x^n)\\
&\frac{1}{1 + x} = 1 - x + x^2 + ... + (-1)^{n-1}x^n + o(x^n)\\
&\tan x = x + \frac{1}{3}x^3 + ... + \frac{(2^{2n} - 1)2^{2n}B_n}{(2n)!}x^{2n-1} + o(x^{2n})\\
&\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + .. + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)} + o(x^{2n})
\end{aligned}$$

Where B_n represents the n-th Bernoulli number.

In fact, the aforementioned formula can be rewritten as mentioned in Taylor formula with infinitesimals of the same order.

[**e.g.9.1.1**]Calculate:

$$\lim_{x\to 0}\frac{\cos x - e^{-\frac{x^2}{2}}}{x^4}$$

Solution:

$$\begin{aligned}
&\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)\\
&e^x = 1 + x + \frac{x^2}{2!} + o(x^2)\\
&e^{-\frac{x^2}{2}} = 1 - \frac{x^2}{2} + \frac{x^4}{8} + o(x^4)
\end{aligned}$$

So that:

$$\begin{aligned}
&\lim_{x\to 0}\frac{\cos x - e^{-\frac{x^2}{2}}}{x^4}\\
&=\lim_{x\to 0}\frac{(1-\frac{x^2}{2!}+\frac{x^4}{4!}+o(x^4))-(1-\frac{x^2}{2}+\frac{x^4}{8}+o(x^4))}{x^4}\\
&=\lim_{x\to 0}\frac{-\frac{x^4}{12}+o(x^4)}{x^4}\\
&=\boxed{-\frac{1}{12}}
\end{aligned}$$

[e.g.9.1.2]Calculate:

$$\lim_{x\to 0}\frac{e^x\sin x - x(1+x)}{x^3}$$

Solution:

$$\begin{aligned}
\sin x &= x-\frac{x^3}{3!}+o(x^4)\\
e^x &= 1+x+\frac{x^2}{2!}+o(x^2)\\
e^x\sin x &= x+x^2+\frac{x^3}{3}+o(x^3)
\end{aligned}$$

So that:

$$\begin{aligned}
&\lim_{x\to 0}\frac{e^x\sin x - x(1+x)}{x^3}\\
&=\lim_{x\to 0}\frac{x+x^2+\frac{x^3}{3}+o(x^3)-x(1+x)}{x^3}\\
&=\lim_{x\to 0}\frac{\frac{x^3}{3}+o(x^3)}{x^3}\\
&=\boxed{\frac{1}{3}}
\end{aligned}$$

textbf[e.g.9.1.3(difficult)]Calculate:

$$\lim_{x\to 0}\frac{e^{(1+x)^{\frac{1}{x}}}-(1+x)^{\frac{e}{x}}}{x^2}$$

Solution:

$$\begin{aligned}
&\lim_{x\to 0}\frac{e^{(1+x)^{\frac{1}{x}}}-(1+x)^{\frac{e}{x}}}{x^2}\\
&=\lim_{x\to 0}((1+x)^{\frac{e}{x}})\cdot\frac{e^{(1+x)^{\frac{1}{x}}-\frac{e}{x}\ln(1+x)}-1}{x^2}\\
&=e^e\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-\frac{e}{x}\ln(1+x)}{x^2}\\
&=e^{e+1}\lim_{x\to 0}\frac{e^{\frac{\ln(1+x)}{x}-1}-\frac{\ln(x+1)}{x}}{x^2}
\end{aligned}$$

Because of

$$\lim_{x\to 0}[\frac{\ln(1+x)}{x}-1]=0$$

So taht

$$e^{\frac{\ln(1+x)}{x}-1}=1+\left(\frac{\ln(1+x)}{x}-1\right)+\frac{1}{2}\left(\frac{\ln(1+x)}{x}-1\right)^2+o\left(\frac{\ln(1+x)}{x}-1\right)^2$$

And

$$\frac{\ln(1+x)}{x}-1=-\frac{x}{2}+\frac{x^2}{3}+o(x^2)$$

So

$$\left(\frac{\ln(1+x)}{x}-1\right)^2=\frac{x^2}{4}+o(x^2)$$

So taht

$$e^{\frac{\ln(1+x)}{x}-1}=1+\left(\frac{\ln(1+x)}{x}-1\right)+\frac{x^2}{8}+o\left(x^2\right)$$

Therefore

$$\begin{aligned}&e^{e+1}\lim_{x\to 0}\frac{e^{\frac{\ln(1+x)}{x}-1}-\frac{\ln(x+1)}{x}}{x^2}\\&=e^{e+1}\lim_{x\to 0}\frac{\frac{\ln(1+x)}{x}+\frac{x^2}{8}+o\left(x^2\right)-\frac{\ln(x+1)}{x}}{x^2}\\&=\boxed{\frac{e^{e+1}}{8}}\end{aligned}$$

9.2 Taylor's Formula with Lagrange's Remainder

Definition 2 If $f(x)$ has n continuous derivatives on $[a,b]$, and is differentiable on(a,b).Then there exists at least one point ξ,such that:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

When $x_0=0$ it is called Maclaurin's formula with Lagrange's remainder. Rewriting the first six terms of the above formula, we obtain:

$$\begin{aligned}&e^x=1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}e^{\theta x}\\&\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}+..+(-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!}+(-1)^m\frac{x^{2m+1}}{(2m+1)!}\cos(\theta x)\\&\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+..+(-1)^m\frac{x^{2m}}{(2m)!}+(-1)^{m+1}\frac{x^{2m+2}}{(2m+2)!}\cos(\theta x)\\&\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+...+(-1)^{n-1}\frac{x^n}{n}+(-1)^n\frac{x^{n+1}}{(n+1)(1+\theta x)^{n+1}}\\&\frac{1}{1-x}=1+x+x^2+...+x^n+\frac{x^{n+1}}{(1-\theta x)^{n+2}}\end{aligned}$$

$$(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+...+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}\cdot(1+\theta x)^{\alpha-n-1}x^{n+1}$$

The above satisfies $0<\theta<1$

9.3 Taylor's Formula with Integral Remainder

Definition: If $f(x)$ has an n+1 th-order derivative at $x = x_0$, and$f^{(n+1)}(x)$ is integrable on $[min\{x, x_0\}, max\{x, x_0\}]$,then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x - t)^n dt$$

If we let $f(x) = e^x, x_0 = 0$, then we have:

$$e^x = 1 + x + \frac{1}{2}x^2 + ... + \frac{x^k}{n!} + \frac{1}{n!}\int_0^x e^t(x - t)^n dt$$

[e.g.9.3.1]Calculate:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \frac{n^k}{k!}}{e^n} = \lim_{n\to\infty} \frac{1 + \frac{n}{1!} + \frac{n^2}{2!} + ... + \frac{n^n}{n!}}{e^n}$$

Solution: Using Maclaurin's formula with the integral remainder term, we obtain:

$$e^n = 1 + \frac{n}{1!} + \frac{n^2}{2!} + ... + \frac{n^n}{n!} + \frac{1}{n!}\int_0^n e^t(n - t)^n dt$$

Therefore:

$$\begin{aligned}&\lim_{n\to\infty} \frac{1 + \frac{n}{1!} + \frac{n^2}{2!} + ... + \frac{n^n}{n!}}{e^n}\\ &= \lim_{n\to\infty} \frac{e^n - \frac{1}{n!}\int_0^n e^t(n - t)^n dt}{e^n}\\ &= 1 - \lim_{n\to\infty} \frac{1}{n!}\frac{\int_0^n e^t(n - t)^n dt}{e^n}\end{aligned}$$

Let $t = nx, dt = ndx$,Then

$$\int_0^n e^t(n - t)^n dt = n^{n+1}\int_0^1 e^{nx}(1 - x)^n dx = n^{n+1}\int_0^1 e^{n[x+ln(1-x)]}dx$$

Using Arzela's Dominated Convergence Theorem, we obtain:

$$\int_0^1 e^{n[x+ln(1-x)]}dx \sim \sqrt{\frac{\pi}{2n}}(n \to \infty)$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n!}\frac{\int_0^n e^t(n - t)^n dt}{e^n} = \lim_{n\to\infty} \frac{n^{n+1}}{n!e^n}\sqrt{\frac{\pi}{2n}}$$

Furthermore, using Stirling's formula $n! \sim \sqrt{2n\pi}(\frac{n}{e})^n$ $(n \to \infty)$,we obtain:

$$\begin{aligned}&\lim_{n\to\infty} \frac{n^{n+1}}{n! \cdot e^n}\sqrt{\frac{\pi}{2n}}\\ &= \lim_{n\to\infty} \frac{n^{n+1}}{\sqrt{2n\pi} \cdot (\frac{n}{e})^n \cdot e^n}\sqrt{\frac{\pi}{2n}}\\ &= \boxed{\frac{1}{2}}\end{aligned}$$

9.4 Taylor's Formula with Cauchy's Remainder

Definition 3 If $f(x)$ has an $n+1$ th-order derivative at $x = x_0$,then there exists $\xi \in [min\{x, x_0\}, max\{x, x_0\}]$, such that:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{1}{n!}f^{(n+1)}(\xi)(x - \xi)^n(x - x_0)dx$$

If we let $\xi = x_0 + \theta(x - x_0), 0 \le \theta \le 1$,then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{1}{n!}f^{(n+1)}(x_0 + \theta(x - x_0))(1 - \theta)^n(x - x_0)^{n+1}dx$$

9.5 Taylor formula with infinitesimals of the same order

Definition 4 **(infinitesimal of the same order)**:If

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$$

and exist$A$,such that

$$|f(x)| \le Ag(x)$$

.Then $f(x)$ and $g(x)$ are said to be infinitesimals of the same order, denoted as

$$f(x) = O(g(x))$$

corollary 1 **(Infinitesimal limit form of the same order)**:If

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$$

and exist$A$,such that

$$\lim_{x \to x_0} \left| \frac{f(x)}{g(x)} \right| \le A$$

.Then $f(x)$ and $g(x)$ are said to be infinitesimals of the same order, denoted as

$$f(x) = O(g(x))$$

So,Taylor's Formula with Peano's RemainderCan be rewritten as Taylor's formula with O:

$$
\begin{aligned}
&e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + O(x^{n+1})\\
&\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + .. + (-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!} + O(x^{2m+1})\\
&\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + .. + (-1)^m\frac{x^{2m}}{(2m)!} + O(x^{2m+2})\\
&\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + (-1)^{n-1}\frac{x^n}{n} + O(x^{n+1})\\
&(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + ... + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + O(x^{n+1})\\
&\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + O(x^{n+1})\\
&\frac{1}{1+x} = 1 - x + x^2 + ... + (-1)^{n-1}x^n + O(x^{n+1})
\end{aligned}
$$

[**e.g.9.5.1**]

$$\lim_{n\to\infty}\frac{1}{n}\left(\frac{n}{\frac{1}{2}+\frac{2}{3}+...+\frac{n}{n+1}}\right)^n$$

Solution:

$$\frac{1}{2}+\frac{2}{3}+...+\frac{n}{n+1} = n - \left(\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n+1}\right) = n + 1 - \gamma_{n+1} - \ln(n+1)$$

Where

$$\gamma_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n+1} - \ln(n+1)$$

.Let

$$a_n = \frac{\gamma_{n+1} + \ln(n+1) - 1}{n}$$

So that:

$$x_n = \frac{1}{n}\left(\frac{n}{\frac{1}{2}+\frac{2}{3}+...+\frac{n}{n+1}}\right) = \frac{1}{n}\left(\frac{1}{1-a_n}\right)^n$$

$$
\begin{aligned}
&\ln x_n = -n\ln(1-a_n) - \ln n\\
&= n(a_n + \frac{{a_n}^2}{2} + ...) - \ln n\\
&= n(a_n + O((\frac{\ln n}{n})^2) - \ln n\\
&= na_n - \ln n + nO((\frac{\ln n}{n})^2)\\
&= \gamma_{n+1} - 1 + \ln\frac{n+1}{n} + nO((\frac{\ln n}{n})^2)
\end{aligned}
$$

Therefore

$$\lim_{n\to\infty}\frac{1}{n}\left(\frac{n}{\frac{1}{2}+\frac{2}{3}+...+\frac{n}{n+1}}\right) = \boxed{e^{\gamma-1}}$$

10 Contraction Mapping

Theorem 1 **(Contractive Mapping for Sequences)**:For any sequence x_n , if there exists $0 < r < 1$ and $N > 0$,such that for all $n > N$,the inequality

$$|x_{n+1} - x_n| \leq r|x_n - x_{n-1}|$$

holds, then the sequence $\{x_n\}$ converges. Such a sequence $\{x_n\}$ is called a contractive sequence.

Proof: Let $|x_{N_0+1} - x_{N_0}| = M$ Then, for any $n > N_0, p \in N$,holds

$$\begin{aligned}
|x_{n+p} - x_n| &= |\sum_{k=1}^{p}(x_{n+k} - x_{n+k-1})| \\
&\leq \sum_{k=1}^{p}|x_{n+k} - x_{n+k-1}| \\
&\leq \sum_{k=1}^{p} r^{n+k-N_0-1}|x_{N_0+1} - x_{N_0}| \\
&= r^{n-N_0}\frac{1-r^p}{1-r}M \\
&\leq \frac{Mr^{n-N_0}}{1-r} \to 0(n \to \infty)
\end{aligned}$$

Therefore, by the Cauchy Convergence Criterion, $\{x_n\}$ converges.

corollary 1 If $x_{n+1} = f(x_n)$,$f(x)$ is differentiable, and there $\exists r \in (0,1), s.t. |f'(x)| \leq r$,then $\{x_n\}$ converges. Proof: Since

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| = |f'(\xi)| \cdot |x_n - x_{n-1}| \leq r|x_n - x_{n-1}|$$

Tt follows from Theorem 1 ,so that$\{x_n\}$ converges.

Definition 1 (Fixed Point)For a function $f(x)$, if there exists $x_0 \in D$, such that

$$f(x_0) = x_0$$

then x_0 is called a fixed point of $f(x)$.

Theorem 2 For a sequence generated by the iteration of a continuous function $f(x)$,i.e.$x_{n+1} = f(x_n)$ if $\{x_n\}$ converges to x_0,then $f(x_0) = x_0$

Proof: Since the limit of $\{x_n\}$exists, we can take the limit on both sides of $x_{n+1} = f(x_n)$ as n approaches infinity, resulting in

$$x_0 = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = \lim_{x\to x_0} f(x) = f(x_0) \quad (Heine'sTheorem)$$

corollary 2 The converse of Theorem 2 is also true: if for all $x \in D$,$f(x) \neq x$,then the sequence $\{x_n\}$ diverges (i.e., its limit does not exist).

Therefore, for some iterative sequences, we can first solve $f(x) = x$ to obtain the limit value x_0, and then substitute $x_0 = f(x_0)$ into the equation we want to prove. That is, to prove $\lim_{n\to\infty} x_{n+1} = x_0$, it suffices to prove that $\lim_{n\to\infty}(x_{n+1} - x_0) = \lim_{n\to\infty}[f(x_n) - f(x_0)] = 0$
.

textbf[e.g.10.1]The sequence $x_0 = 1$,$x_{n+1} = \sqrt{2x_n}$ has a limit. Calculate,Calculate $\lim_{n\to\infty} x_n$

Solution: Because of $1 \le x_0 < 2$, we assume $1 \le x_n < 2$, then $1 \le x_{n+1} = \sqrt{2x_n} < 2$ holds..

Method 1: Let

$$f(x) = \sqrt{2x}, |f'(x)| = \frac{\sqrt{2}}{2\sqrt{x}} \le \frac{\sqrt{2}}{2} < 1$$

By Corollary 1,$\{x_n\}$converges. Taking the limit on both sides of $x_{n+1} = \sqrt{2x_n}$ as n approaches infinity, we get $x = \sqrt{2x}$,which yields $x = 2$.

Method 2: Taking the limit on both sides of $x_{n+1} = \sqrt{2x_n}$ as n approaches infinity, we get $x = \sqrt{2x}$, which yields $x = 2$. Then,

$$|x_n - 2| = |\frac{x_{n+1}^2}{2} - \frac{2^2}{2}| = \frac{1}{2}|x_{n+1} - 2| \cdot |x_{n+1} + 2| \ge \frac{3}{2}|x_{n+1} - 2|$$

Let $q = \frac{2}{3}$ then

$$|x_{n+1} - 2| \le \frac{2}{3}|x_n - 2| = q|x_n - 2| \le q^2|x_{n-1} - 2| \le ... \le q^n|x_1 - 2| \to 0(n \to 0)$$

Therefore,$\lim_{n\to\infty} x_n = 2$

[e.g.10.2(Difficult)](1) Let $f_1(t) = \frac{t+3}{2}$,$f_2(t) = \frac{t+6}{3}$,and $\{n_k\}$ be an integer sequence taking values in $\{1, 2\}$. Define $F_1(t) = f_{n_1}(t)$,$F_{k+1}(t) = F_k(f_{n_{k+1}}(t))(k \ge 1)$. Prove that for any $x \in R$, the limit $\lim_{k\to\infty} F_k(x)$ exists and is independent of x .
(2) What is the conclusion if f_1, f_2 in problem (1) are replaced by $f_1(t) = t - \arctan(t)$ and $f_2(t) = 2\arctan(t) - t$?

Solution:(1) Proof: Since $f_1'(t) = \frac{1}{2}$, $f_2'(t) = \frac{1}{3}$, we have

$$|f_k(t) - f_k(s)| \le \frac{1}{2}|t - s| \quad (k = 1, 2)$$

Therefore,f_1, f_2 are contraction mappings. Moreover, they have a unique common fixed point $x_0 = 3$

So taht

$$\begin{aligned}
&|F_k(x) - 3| = |f_{n_1} \circ f_{n_2} \circ ... \circ f_{n_k}(x) - f_{n_1}(3)\\
&\le \frac{1}{2}|f_{n_2} \circ f_{n_3} \circ ... \circ f_{n_k}(x) - 3|\\
&= \frac{1}{2}|f_{n_2} \circ f_{n_3} \circ ... \circ f_{n_k}(x) - f_{n_2}(3)|\\
&\le ...\\
&\le \frac{1}{2^{k-1}}|f_{n_k}(x) - 3|\\
&= \frac{1}{2^{k-1}}|f_{n_k}(x) - f_{n_k}(3)|\\
&\le \frac{1}{2^k}|x - 3|
\end{aligned}$$

From this, we can conclude that $\lim_{k\to\infty} F_k(x) = 3$

(2) We conjecture that when both $f_1(t)$ and $f_2(t)$share the same fixed point and are contraction mappings, the calculated limit exists and is independent of x. Given that

$$f_1'(t) = \frac{t^2}{1+t^2}, f_2'(t) = \frac{1-t^2}{1+t^2}$$

Let $|f_k'(t)| \leq 1$,then $t_0 = 0$,and $|f_k(t)| \leq |t|, f_k(t_0) = 0$ Next, we prove that

$$\lim_{k\to\infty} F_k(x) = 0$$

Otherwise, there would exist a subsequence $\{F_{m_k}(x)\}$ of $\{F_k(x)\}$,such that $0 < \delta < |F_{m_k}(x)| \leq |x|$ At this point, $|f_1'|, |f_2'|$ have an upper bound l on the intervals $[-|x|, \frac{\delta}{2}] \bigcup [\frac{\delta}{2}, |x|]$. For any $y \in [\delta, |x|]$, we have

$$\begin{aligned}
&|f_k(\pm y)| = |f_k(\pm y) - f_k(\pm\frac{\delta}{2}) + f_k(\pm\frac{\delta}{2}) - f_k(0)| \\
&\leq |f_k(\pm y) - f_k(\pm\frac{\delta}{2})| + |f_k(\pm\frac{\delta}{2}) - f_k(0)| \\
&\leq l(y - \frac{\delta}{2}) + \frac{\delta}{2} = ly + (1-l)\frac{\delta}{2} \\
&\leq ly + (1-l)\frac{y}{2} \\
&= \frac{l+1}{2} y
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\delta \leq |F_{m_k}(x)| = |f_{n_1} \circ f_{n_2} \circ ... \circ f_{n_{m_k}}| \\
&\leq |f_{n_j} \circ f_{n_{j+1}} \circ ... \circ f_{n_{m_k}}| \\
&\leq |x| (j = 1, 2 ... m_k)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|F_{m_k}(x)| = |f_{n_1} \circ f_{n_2} \circ ... \circ f_{n_{m_k}}| \\
&\leq \frac{l+1}{2} |f_{n_2} \circ ... \circ f_{n_{m_k}}| \\
&\leq \left(\frac{l+1}{2}\right)^{m_k} |x|
\end{aligned}$$

Letting $k \to +\infty$ we obtain $\lim_{k\to+\infty} |F_{m_k}| = 0$. This is a contradiction. Therefore, $\lim_{k\to+\infty} |F_k| = 0$ □

11 Stolz's Theorem

Stolz's Theorem, also known as Stolz's Formula, is an effective method for finding the limits of certain sequences, akin to L'Hôpital's rule for sequences (see L'Hôpital's rule for functions in L'Hospital's Rule).

11.1 Stolz's Formula for Sequences

Theorem 1 ($\frac{0}{0}$ Indeterminate Form) If the sequences a_n and b_n satisfy: b_n is monotonically decreasing and converges to 0, a_n converges to 0, and

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=A$$

then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=A$$

where A can be $+\infty$ or $-\infty$.

Theorem 2 ($\frac{*}{\infty}$ Indeterminate Form) If the sequences a_n and b_n satisfy: b_n is monotonically increasing and converges to ∞, and

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=A$$

then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=A$$

where A can be $+\infty$ or $-\infty$.

Now let's revisit Example 1.1.4 using Stolz's Theorem. The limit satisfies Theorem 2, thus:

$$\lim_{n\to\infty}\frac{a_1+a_2+...+a_n}{n}=\lim_{n\to\infty}\frac{a_n}{1}=a$$

[**e.g.11.1.1(Difficult)**] Let $S_n=\dfrac{\sum_{k=0}^{n}\ln C_n^k}{n^2}$, find $\lim_{n\to\infty}S_n$

Solution: Using Stolz's formula, we have:

$$\begin{aligned}\lim_{n\to\infty}S_n&=\lim_{n\to\infty}\frac{\sum_{k=0}^{n}\ln C_n^k}{n^2}\\&=\lim_{n\to\infty}\frac{\sum_{k=0}^{n+1}\ln C_{n+1}^k-\sum_{k=0}^{n}\ln C_n^k}{(n+1)^2-n^2}\\&=\lim_{n\to\infty}\frac{\sum_{k=0}^{n}\ln\frac{C_{n+1}^k}{C_n^k}-\ln C_{n+1}^{n+1}}{2n+1}\\&=\lim_{n\to\infty}\frac{\sum_{k=0}^{n}\ln\frac{n+1}{n-k+1}}{2n+1}\end{aligned}$$

Since

$$\sum_{k=0}^{n}\ln(n-k+1)=\sum_{k=1}^{n+1}\ln(k)=\ln(n+1)!$$

therefore:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \ln \frac{n+1}{n-k+1}}{2n+1} \\
&= \lim_{n\to\infty} \frac{(n+1)\ln(n+1) - \sum_{k=1}^{n+1} \ln(k)}{2n+1} \\
&= \lim_{n\to\infty} \frac{(n+1)\ln(n+1) - n\ln n - \ln(n+1)}{2} \\
&= \lim_{n\to\infty} \frac{1}{2} \cdot \ln\left(\left(\frac{n+1}{n}\right)^n\right) \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

[e.g. 11.1.2] Suppose $\lim\limits_{n\to\infty} n(A_n - A_{n-1}) = 0$

Prove that when $\lim_{n\to\infty} \frac{A_1+A_2+...+A_n}{n}$ exists, we have

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{A_1 + A_2 + ... + A_n}{n}$$

Proof: Let $a_1 = A_1$ and $a_n = A_n - A_{n-1}$ for $n \geq 2$. Then

$$\begin{aligned}
&A_n = \left(A_n - \frac{A_1 + A_2 + ... + A_n}{n}\right) + \frac{A_1 + A_2 + ... + A_n}{n} \\
&A_n = (A_n - A_{n-1}) + (A_{n-1} - A_{n-2}) + ... + (A_2 - A_1) + A_1 \\
&= a_n + a_{n-1} + ... + a_1 \\
&\lim_{n\to\infty} n(A_n - A_{n-1}) = \lim_{n\to\infty} na_n = 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} \left(A_n - \frac{A_1 + A_2 + ... + A_n}{n}\right) \\
&= \lim_{n\to\infty} \left[(a_1 + a_2 + ... + a_n) - \frac{na_1 + (n-1)a_2 + ... + a_n}{n}\right] \\
&= \lim_{n\to\infty} \frac{a_2 + 2a_3 + ... + (n-1)a_n}{n} \\
&= \lim_{n\to\infty} \frac{(n-1)a_n}{1} \\
&= 0
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{A_1 + A_2 + ... + A_n}{n}$$

[e.g. 11.1.3] Let $x_0 = a$, where $0 < a < \frac{\pi}{2}$, and $x_n = \sin x_{n-1}$ for $n = 1, 2, ...$ Prove that:

$$\lim_{n\to\infty} \sqrt{\frac{n}{3}} x_n = 1$$

Proof: First, we prove that $\lim_{n\to\infty} x_n = 0$ (otherwise the limit we seek does not exist). Since $0 < a < \frac{\pi}{2}$ and $x_0 = a$, we have

$$0 < x_n = \sin x_{n-1} < x_{n-1} < \frac{\pi}{2} \quad \text{for} \quad n = 1, 2, ...$$

Therefore, x_n is monotonically decreasing and bounded below, so $\lim_{n\to\infty} x_n$ exists. Let this limit be x, then $x = \sin x$, so $x = 0$, i.e., $\lim_{n\to\infty} x_n = 0$.

Next, we only need to prove that

$$\lim_{n\to\infty} \frac{n}{\frac{1}{x_n^2}} = 3$$

Since

$$\begin{aligned}
&\lim_{n\to\infty} \frac{n}{\frac{1}{x_n^2}} \\
&= \lim_{n\to\infty} \frac{(n+1)-n}{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}} \\
&= \lim_{n\to\infty} \frac{x_n^2 \sin^2 x_n}{x_n^2 - \sin^2 x_n} \\
&= \lim_{x\to 0} \frac{x^2 \sin^2 x}{x^2 - \sin^2 x} \\
&= \lim_{x\to 0} \frac{x^4}{(x+\sin x)(x-\sin x)} \\
&= \lim_{x\to 0} \frac{x^4}{(2x+o(x))(\frac{x^3}{6}+o(x^3))} \\
&= 3
\end{aligned}$$

Therefore $\lim_{n\to\infty} \sqrt{\frac{n}{3}} x_n = 1$ □

11.2 Stolz's Theorem in Functional Form

Theorem 3 ($\frac{0}{0}$ Indeterminate Form) Suppose $T > 0$ and the following conditions are satisfied:
(1) $0 < g(x+T) < g(x)$;
(2) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} g(x) = 0$;
(3) $\lim\limits_{x\to+\infty} \dfrac{f(x+T)-f(x)}{g(x+T)-g(x)} = l$.
Then $\lim\limits_{x\to+\infty} \dfrac{f(x)}{g(x)} = l$, where l can be $+\infty, -\infty$.

Theorem 4 ($\frac{*}{\infty}$ Indeterminate Form) Suppose $T > 0$ and the following conditions are satisfied:
(1) $g(x+T) > g(x)$;
(2) $\lim\limits_{x\to+\infty} g(x) = +\infty$, and $f(x), g(x)$ are bounded on any closed interval within $[a, +\infty)$;
(3) $\lim\limits_{x\to+\infty} \dfrac{f(x+T)-f(x)}{g(x+T)-g(x)} = l$.
Then $\lim\limits_{x\to+\infty} \dfrac{f(x)}{g(x)} = l$, where l can be $+\infty, -\infty$.

[**e.g.11.2.1**] Suppose $f(x)$ is defined on $[a, +\infty)$ and is bounded on any closed interval within this range. Given that $\lim_{x\to+\infty} \frac{f(x+1)-f(x)}{x^n} = l$, prove that:

$$\lim_{x\to+\infty} \frac{f(x)}{x^{n+1}} = \frac{l}{n+1}$$

Proof:

$$\begin{aligned}
&\lim_{x\to+\infty}\frac{f(x)}{x^{n+1}}\\
&=\lim_{x\to+\infty}\frac{f(x+1)-f(x)}{(x+1)^{n+1}-x^{n+1}}\\
&=\lim_{x\to+\infty}\frac{f(x+1)-f(x)}{(n+1)x^n+\frac{(n+1)n}{1\cdot2}x^{n-1}+\ldots+1}\\
&=\lim_{x\to+\infty}\frac{\frac{f(x+1)-f(x)}{x^n}}{(n+1)+\frac{(n+1)n}{1\cdot2}\frac{1}{x}+\ldots+\frac{1}{x^n}}\\
&=\frac{l}{n+1}
\end{aligned}$$

This completes the proof. □

11.3 Converse Theorem of Stolz's Theorem

The converse theorem of Stolz's Theorem does not necessarily hold. For instance, take $x_n=(-1)^n, y_n=n$. Although $\lim_{n\to\infty}\frac{x_n}{y_n}=0$, the limit $\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}$ does not exist.

corollary 1 In fact, if x_n, y_n satisfy the conditions of Stolz's Theorem, and both $\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}$ and $\lim_{n\to\infty}\frac{x_n}{y_n}$ exist, then we only need to know $\lim_{n\to\infty}\frac{x_n}{y_n}$. By applying Stolz's Theorem, we immediately obtain:

$$\lim_{n\to\infty}\frac{x_n}{y_n}=\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}$$

However, to use this conclusion, we need to know or prove beforehand that the limit $\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}$ exists.

The following theorem does not require prior knowledge of $\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}$, but adds some other conditions. Starting from the properties of limits, it determines whether the converse theorem of Stolz's Theorem holds.

Theorem 5 ($\frac{*}{\infty}$ Form) If $\lim_{n\to\infty}y_n=+\infty$, and there exist $N>0, A\in\mathbb{R}$ such that when $n>N$, $y_{n+1}>y_n$, then:
(1) If $\lim_{n\to\infty}\frac{y_n}{y_n-y_{n-1}}=A$ and $\lim_{n\to\infty}\frac{x_n}{y_n}=l$, then

$$\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}=l$$

(2) If $\lim_{n\to\infty}\frac{x_n}{x_n-x_{n-1}}=A$ and $\lim_{n\to\infty}\frac{x_n}{y_n}=l$, then

$$\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}=l$$

Note: l can be $+\infty, -\infty$.

[e.g.11.3.1] Solve for:

$$\lim_{n\to\infty}\frac{n^k-(n-1)^k}{e^n-e^{n-1}}$$

Solution: First,

$$\lim_{n\to\infty}\frac{n^k}{e^n}=\left(\lim_{n\to\infty}\frac{n}{e^{\frac{n}{k}}}\right)^k=\left(\lim_{n\to\infty}\frac{k}{e^{\frac{n}{k}}\cdot\frac{1}{n}}\right)^k\cdot\lim_{n\to\infty}\frac{1}{n^0}=0^k=0$$

(Note: Here we used the fact that $\lim_{n\to\infty} \frac{1}{n} = 0$ and the property of limits.)

Since it satisfies the conditions of Theorem 5, we have:

$$\lim_{n\to\infty} \frac{n^k - (n-1)^k}{e^n - e^{n-1}} = \lim_{n\to\infty} \frac{n^k}{e^n} = \boxed{0}$$

Theorem 6 ($\frac{0}{0}$ Form) If $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = 0$, and there exist $N > 0, A \in \mathbb{R}$ such that when $n > N$, $y_{n+1} < y_n$, then:

(1) If $\frac{y_n}{y_n - y_{n-1}}$ is bounded above and $\lim_{n\to\infty} \frac{x_n}{y_n} = A$, then

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = A$$

(2) If $\frac{x_n}{x_n - x_{n-1}}$ is bounded above and $\lim_{n\to\infty} \frac{x_n}{y_n} = \infty$, then

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \infty$$

Theorem 7 (Stolz's Converse Theorem in Functional Form) Let $T > 0, A, l \in \mathbb{R}$, and suppose $g(x+T) > g(x)$ and $\lim_{x\to+\infty} g(x) = +\infty$. If:

(1) $\lim_{x\to+\infty} \frac{g(x)}{g(x) - g(x+T)} = A$ and $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = l$, then

$$\lim_{x\to+\infty} \frac{f(x+T) - f(x)}{g(x+T) - g(x)} = l$$

(2) $\lim_{x\to+\infty} \frac{f(x)}{f(x) - f(x+T)} = A$ and $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \infty$, then

$$\lim_{x\to+\infty} \frac{f(x+T) - f(x)}{g(x+T) - g(x)} = \infty$$

Theorem 8 Let $T > 0, l \in \mathbb{R}$, and suppose $g(x+T) < g(x)$ and $\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} f(x) = 0$. If:

(1) $\lim_{x\to+\infty} \frac{g(x)}{g(x) - g(x+T)}$ is bounded above and $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = l$, then

$$\lim_{x\to+\infty} \frac{f(x+T) - f(x)}{g(x+T) - g(x)} = l$$

(2) $\lim_{x\to+\infty} \frac{f(x)}{f(x) - f(x+T)}$ is bounded above and $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \infty$, then

$$\lim_{x\to+\infty} \frac{f(x+T) - f(x)}{g(x+T) - g(x)} = \infty$$

12 Mean Value Theorems

12.1 Differential Mean Value Theorems

When solving limit problems, we often encounter structures in limits that involve the difference of function values of one or two functions. In such cases, we can consider using the Lagrange Mean Value Theorem and the Cauchy Mean Value Theorem to express the function difference in terms of derivatives. Sometimes, this manipulation can greatly simplify the calculations and make it easier to find the limit.

12.1.1 Rolle's Mean Value Theorem

Theorem 1 Suppose $f(x)$ satisfies:
(1) It is continuous on the closed interval $[a, b]$;
(2) It is differentiable on the open interval (a, b);
(3) $f(a) = f(b)$.
Then there exists at least one point ξ in (a, b) such that $f'(\xi) = 0$.

12.1.2 Lagrange's Mean Value Theorem

Theorem 2 Suppose $f(x)$ satisfies:
(1) It is continuous on the closed interval $[a, b]$;
(2) It is differentiable on the open interval (a, b).
Then there exists at least one point ξ in (a, b) such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.

[e.g.12.1.2.1] Find:

$$\lim_{n\to\infty} n^2\left(\arctan\frac{1}{n} - \arctan\frac{1}{n+1}\right)$$

Solution:

$$\begin{aligned}
&\lim_{n\to\infty} n^2\left(\arctan\frac{1}{n} - \arctan\frac{1}{n+1}\right)\\
&= \lim_{n\to\infty} n^2 \cdot \frac{1}{1+\xi^2}\cdot\left(\frac{1}{n} - \frac{1}{n+1}\right) \quad \left(\xi \in \left(\frac{1}{n+1}, \frac{1}{n}\right)\right)\\
&= \lim_{n\to\infty} n^2 \cdot \left(\frac{1}{n} - \frac{1}{n+1}\right)\\
&= \boxed{1}
\end{aligned}$$

[e.g.12.1.2.2] Find:

$$\lim_{x\to 0}\frac{e^x - e^{\sin x}}{x^2\ln(1+x)}$$

Solution:

$$\begin{aligned}
&\lim_{x\to 0}\frac{e^x - e^{\sin x}}{x^2\ln(1+x)}\\
&= \lim_{x\to 0}\frac{e^{\xi}(x-\sin x)}{x^2\ln(1+x)} \quad (\xi \text{ is between } x \text{ and } \sin x)\\
&= \lim_{x\to 0}\frac{x - \left(x - \frac{x^3}{6} + o(x^3)\right)}{x^3}\\
&= \boxed{\frac{1}{6}}
\end{aligned}$$

12.1.3 Cauchy's Mean Value Theorem

Theorem 3 Suppose $f(x), g(x)$ satisfy:
(1) Continuous on the closed interval $[a, b]$;
(2) Differentiable on the open interval (a, b);
(3) $f'(x), g'(x)$ are not zero simultaneously;
(4) $g(a) \neq g(b)$.
Then there exists at least one point ξ in (a, b) such that $\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

[e.g.12.1.3.1] Find:

$$\lim_{x\to 2} \frac{\sin(x^x) - \sin(2^x)}{2^{x^x} - 2^{2^x}}$$

Solution: Let $f(x) = \sin x, g(x) = 2^x$, then

$$\begin{aligned}
&\lim_{x\to 2} \frac{\sin(x^x) - \sin(2^x)}{2^{x^x} - 2^{2^x}} \\
&= \lim_{x\to 2} \frac{\cos\xi \cdot (x^x \ln x - 2^x \ln 2)}{2^\xi \cdot (\xi \ln 2)} \quad (\xi \text{ is between } \min\{x^x, 2^x\} \text{ and } \max\{x^x, 2^x\}) \\
&= \boxed{\frac{\cos 4}{16 \ln 2}}
\end{aligned}$$

[e.g.12.1.3.2] Find:

$$\lim_{x\to 0} \frac{e^{ax} - e^{bx}}{\sin ax - \sin bx}$$

Solution:

$$\begin{aligned}
&\lim_{x\to 0} \frac{e^{ax} - e^{bx}}{\sin ax - \sin bx} \\
&= \lim_{x\to 0} \frac{ae^\xi - be^\xi}{\cos\xi (a-b)x} \quad (\xi \text{ is between } ax \text{ and } bx) \\
&= \lim_{x\to 0} \frac{e^\xi (a-b)}{\cos\xi (a-b)x/x} \\
&= \lim_{x\to 0} \frac{e^\xi}{\cos\xi} \\
&= \boxed{1}
\end{aligned}$$

12.2 Mean Value Theorems for Integrals

The Mean Value Theorems for Integrals have many similarities to the Mean Value Theorems for Derivatives. If a definite integral with variable limits is viewed as a function $f(x)$, then the Mean Value Theorem for Derivatives (when conditions are met) can also be applied to $f(x)$. However, there are also some differences, especially in the form of the Second Mean Value Theorem for Integrals.

12.2.1 First Mean Value Theorem for Integrals

Theorem 4 Suppose $f(x)$ is continuous on the closed interval $[a, b]$, then there exists at least one point $\xi \in [a, b]$ such that:

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

Theorem 5 Suppose $f(x), g(x)$ are continuous on the closed interval $[a, b]$, and $g(x)$ does not change sign on $[a, b]$, then there exists at least one point $\xi \in [a, b]$ such that:

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$$

Theorem 6 Suppose $f(x, y)$ is continuous on a bounded closed region D, then there exists at least one point $(\xi, \eta) \in D$ such that:

$$\iint_D f(x, y)d\sigma = f(\xi, \eta) \cdot \sigma_0$$

where σ_0 is the area of D.

Theorem 7 Suppose $f(x, y), g(x, y)$ are continuous on a bounded closed region D, and $g(x, y)$ does not change sign on D, then there exists at least one point $(\xi, \eta) \in D$ such that:

$$\iint_D f(x, y)g(x, y)d\sigma = f(\xi, \eta)\iint_D g(x, y)d\sigma$$

[e.g.12.2.1.1] Find:

$$\lim_{n\to\infty}\int_0^1 \sin^n x dx$$

Solution: According to the Mean Value Theorem for Integrals, there exists $\xi_n \in [0, 1]$ such that:

$$\int_0^1 \sin^n x dx = (1-0)\sin^n \xi_n = \sin^n \xi_n$$

Since $\xi_n \in [0, 1]$, it follows that $0 \le \sin^n \xi_n \le \sin^n 1 \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty}\int_0^1 \sin^n x dx = 0$$

□

[e.g.12.2.1.2] Find:

$$\lim_{x\to 0}\int_x^{3x} \frac{\cos t}{t} dt$$

Solution: According to the Mean Value Theorem for Integrals, there exists $\xi \in [x, 3x]$ (note: corrected from ξ_n to ξ as it's not indexed by n here) such that:

$$\begin{aligned}
&\lim_{x\to 0}\int_x^{3x} \frac{\cos t}{t} dt \\
&= \lim_{x\to 0}\cos\xi \int_x^{3x} \frac{1}{t} dt \\
&= \lim_{x\to 0}\cos\xi \ln\left(\frac{3x}{x}\right) \\
&= \lim_{x\to 0}\cos\xi \ln 3 \\
&= \boxed{\ln 3}
\end{aligned}$$

[e.g.12.2.1.3] Find:

$$\lim_{x\to 0}\frac{\int_{\sin x}^{x}\sin t^2 dt}{x^5}$$

Solution: According to the Mean Value Theorem for Integrals, there exists $\xi \in [\sin x, x]$ such that:

$$\begin{aligned}
&\lim_{x\to 0}\frac{\int_{\sin x}^{x}\sin t^2 dt}{x^5}\\
&=\lim_{x\to 0}\frac{(x-\sin x)\sin\xi^2}{x^5}\\
&=\lim_{x\to 0}\frac{\left(\frac{x^3}{6}+o(x^3)\right)\sin\xi^2}{x^5}\\
&=\frac{1}{6}\lim_{x\to 0}\left(\frac{\xi}{x}\right)^2\sin\xi^2\\
&=\frac{1}{6}\lim_{x\to 0}\left(\frac{\xi}{x}\right)^2\\
&=\boxed{\frac{1}{6}}
\end{aligned}$$

12.2.2 The Second Mean Value Theorem for Integrals

Theorem 8 Suppose $f(x)$ and $g(x)$ are integrable on the closed interval $[a, b]$, and $f(x)$ is a monotonic function. Then there exists at least one point $\xi \in [a, b]$ such that:

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{\xi} g(x)dx + f(b)\int_{\xi}^{b} g(x)dx$$

Theorem 9 Suppose $f(x)$ and $g(x)$ are integrable on the closed interval $[a, b]$, $f(x) \geq 0$ and $f(x)$ is monotonically decreasing. Then there exists at least one point $\xi \in [a, b]$ such that:

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{\xi} g(x)dx$$

Theorem 10 Suppose $f(x)$ and $g(x)$ are integrable on the closed interval $[a, b]$, $f(x) \geq 0$ and $f(x)$ is monotonically increasing. Then there exists at least one point $\xi \in [a, b]$ such that:

$$\int_a^b f(x)g(x)dx = f(b)\int_{\xi}^{b} g(x)dx$$

[e.g.12.2.2.1(Difficult)]: Find:

$$\lim_{n\to\infty}\int_n^{n+1}(\cos x^2)^2dx$$

Solution: Let $t = x^2$ and $dx = \frac{1}{2\sqrt{t}}dt$, then:

$$\begin{aligned}
I &= \int_n^{n+1}(\cos x^2)^2dx \\
&= \int_{n^2}^{(n+1)^2}(\cos t)^2\cdot\frac{1}{2\sqrt{t}}dt \\
&= \int_{n^2}^{(n+1)^2}\frac{1+\cos 2t}{2}\cdot\frac{1}{2\sqrt{t}}dt \\
&= \int_{n^2}^{(n+1)^2}\frac{1}{4\sqrt{t}}dt + \int_{n^2}^{(n+1)^2}\frac{\cos 2t}{4\sqrt{t}}dt \\
&= I_1 + I_2
\end{aligned}$$

For I_1:

$$I_1 = \int_{n^2}^{(n+1)^2}\frac{1}{4\sqrt{t}}dt = \left.\frac{\sqrt{t}}{2}\right|_{n^2}^{(n+1)^2} = \frac{1}{2}$$

For I_2:

$$I_2 = \int_{n^2}^{(n+1)^2}\frac{\cos 2t}{4\sqrt{t}}dt = \frac{1}{4n}\int_{n^2}^{\xi}\cos 2t\,dt + \frac{1}{4(n+1)}\int_{\xi}^{(n+1)^2}\cos 2t\,dt$$

(where ξ is some point between n^2 and $(n+1)^2$)

And

$$I_2 \ge \frac{1}{4(n+1)}\int_{n^2}^{(n+1)^2}\cos 2t\,dt = \frac{\sin 2(n+1)^2 - \sin 2n^2}{8(n+1)} \ge -\frac{1}{4(n+1)}$$

$$I_2 \le \frac{1}{4n}\int_{n^2}^{(n+1)^2}\cos 2t\,dt = \frac{\sin 2(n+1)^2 - \sin 2n^2}{8n} \le \frac{1}{4n}$$

Therefore,

$$\frac{1}{2} - \frac{1}{4(n+1)} \le \int_n^{n+1}(\cos x^2)^2dx \le \frac{1}{2} + \frac{1}{4n}$$

Hence,

$$\lim_{n\to\infty}\int_n^{n+1}(\cos x^2)^2dx = \boxed{\frac{1}{2}}$$

13 Definition of Definite (Riemann) Integral

We often encounter a type of integral that involves summation, and sometimes the Sandwich Theorem (Squeeze Theorem) may not be applicable. In such cases, we can consider using the definition of the definite integral to solve the integral. According to the definition of the definite integral, we can also solve a type of integral that involves summation.

13.1 Definition of Definite Integral

Definition 1 Let there be $n-1$ points on the closed interval $[a, b]$, denoted as:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

Each small interval is denoted as

$$\Delta_i = [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n.$$

These division points or intervals are called a partition of $[a, b]$, denoted as $T = \{x_0, x_1, \ldots, x_n\}$ or $T = \{\Delta_1, \Delta_2, \ldots, \Delta_n\}$. The length of each small interval is denoted as

$$\Delta x_i = x_i - x_{i-1}$$

and

$$||T|| = \max_{1 \le i \le n} \{\Delta x_i\}$$

is called the norm of T.

Definition 2 Let $f(x)$ be a function defined on $[a, b]$, and J be a real number. If for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition T of $[a, b]$ and any $\xi_i \in \Delta_i$, as long as $||T|| < \delta$, we have

$$\left| \sum_{i=1}^{n} f(\xi_i) \Delta x_i - J \right| < \varepsilon$$

then $f(x)$ is said to be Riemann integrable on $[a, b]$, and J is called the definite integral of $f(x)$ on $[a, b]$, denoted as:

$$J = \int_a^b f(x) dx$$

corollary 1 If $f(x)$ is Riemann integrable on $[a, b]$, a common partition is $\Delta_i = \frac{b-a}{n}$. Then, according to the definition of the definite integral, we have:

$$\lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)$$

[e.g.13.1.1]Find:

$$\lim_{n \to \infty} \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}} \quad (k \ge 0)$$

Solution:

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1^k+2^k+...+n^k}{n^{k+1}}\\
&=\lim_{n\to\infty}\frac{1}{n}\frac{\sum_{i=1}^{n}i^k}{n^k}\\
&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(\frac{i}{n})^k\\
&=\int_0^1 x^k dx\\
&=\boxed{\frac{1}{k+1}}
\end{aligned}$$

[e.g.13.1.2] Find:

$$\lim_{n\to\infty}\frac{\sqrt{n!}}{n}$$

Solution:

$$\frac{\sqrt{n!}}{n}=e^{\frac{\ln n!}{n}-\ln n}=e^{\frac{1}{n}\ln\frac{1}{n}+\ln\frac{2}{n}+...+\ln\frac{n}{n}}$$

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\sqrt{n!}}{n}\\
&=e^{\lim_{n\to\infty}\frac{1}{n}(\ln\frac{1}{n}+\ln\frac{2}{n}+...+\ln\frac{n}{n})}
\end{aligned}$$

And

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}(\ln\frac{1}{n}+\ln\frac{2}{n}+...+\ln\frac{n}{n})\\
&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\frac{k}{n}\\
&=\int_0^1\ln x dx\\
&=(x\ln x-x)|_0^1\\
&=-1
\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\frac{\sqrt{n!}}{n}=\boxed{\frac{1}{e}}$$

[e.g.13.1.3(Difficult)] If $f(x)$ has a continuous second derivative on $[a, b]$, prove that:

$$\lim_{n\to\infty} n\left[\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)-\int_a^b f(x)dx\right]=\frac{b-a}{2}[f(b)-f(a)]$$

Proof: Expand $f(x)$ at

$$x_k = a + k\frac{b-a}{n}$$

using Taylor's formula with Lagrange remainder, then there exists

$$\xi_k \in \left(a+(k-1)\frac{b-a}{n}, a+k\frac{b-a}{n}\right) \quad (k=1,2,...,n)$$

such that

$$f(x) = f(x_k) + f'(x_k)(x-x_k) + \frac{f''(\xi_k)}{2}(x-x_k)^2$$

Since

$$(b-a)\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right) = \sum_{k=1}^{n} n\int_{a+(k-1)\frac{b-a}{n}}^{a+k\frac{b-a}{n}} f(x_k)dx$$

$$n\int_a^b f(x)dx = \sum_{k=1}^{n} n\int_{a+(k-1)\frac{b-a}{n}}^{a+k\frac{b-a}{n}} f(x)dx$$

Since $f(x)$ has a continuous second derivative on $[a, b]$, then

$$\exists M = \max|f''([a,b])|$$

and

$$\begin{aligned}
&\left|(b-a)\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right) - n\int_a^b f(x)dx + \sum_{k=1}^{n} n\int_{a+(k-1)\frac{b-a}{n}}^{a+k\frac{b-a}{n}} f'(x_k)(x-x_k)dx\right| \\
&= \left|\sum_{k=1}^{n} n\int_{a+(k-1)\frac{b-a}{n}}^{a+k\frac{b-a}{n}} [f(x_k)+f'(x_k)(x-x_k)-f(x)]dx\right| \\
&= \left|\sum_{k=1}^{n} n\int_{a+(k-1)\frac{b-a}{n}}^{a+k\frac{b-a}{n}} \frac{f''(\xi_k)}{2}(x-x_k)^2dx\right| \\
&\le \frac{M}{2}\left|\sum_{k=1}^{n} n\int_{a+(k-1)\frac{b-a}{n}}^{a+k\frac{b-a}{n}} (x-x_k)^2dx\right| \\
&= \frac{M}{6}\left|\sum_{k=1}^{n} n(x-x_k)^3 \Big|_{a+(k-1)\frac{b-a}{n}}^{a+k\frac{b-a}{n}}\right| \\
&= \frac{M(b-a)^3}{6n} \to 0 \quad (n\to\infty)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{n\to\infty}\left[(b-a)\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)-n\int_a^b f(x)dx\right]\\
&=-\lim_{n\to\infty}\left[\sum_{k=1}^{n} n\int_{a+(k-1)\frac{b-a}{n}}^{a+k\frac{b-a}{n}} f'(x_k)(x-x_k)dx\right]\\
&=-\lim_{n\to\infty}\frac{1}{2}\sum_{k=1}^{n} nf'(x_k)(x-x_k)^2\ |_{a+(k-1)\frac{b-a}{n}}^{a+k\frac{b-a}{n}}\\
&=\lim_{n\to\infty}\frac{(b-a)^2}{2n}\sum_{k=1}^{n} f'\left(a+k\frac{b-a}{n}\right)\\
&=\frac{b-a}{2}\int_a^b f'(x)dx\\
&=\boxed{\frac{b-a}{2}\left[f(b)-f(a)\right]}
\end{aligned}$$

Note: If $f(x)$ has a continuous $p+1$th derivative on $[a,b]$, then this problem can be extended to the following form:

$$\lim_{n\to\infty} n^p\left\{\left[\sum_{k=0}^{p-1}(-1)^k\cdot\frac{(b-a)^{k+1}}{k!\cdot(k+1)n^{k+1}}\sum_{i=1}^{n} f^{(k)}(x_i)\right]-\int_a^b f(x)dx\right\}=\frac{(-1)^{p+1}(b-a)^p}{(p+1)!}\left[f^{(p-1)}(b)-f^{(p-1)}(a)\right]$$

Due to the cumbersome proof, it is omitted here. For a detailed proof, see https://zhuanlan.zhihu.com/p/931322500.

13.2 Definition of Double Integral

Imitating the definition of the definite integral, we can similarly define the double integral.

Definition 3 Divide the region D into n small regions σ_i (for $i=1,2,...,n$), and denote the area of each small region as σ_i as well. These small regions are called a partition of D, denoted as

$$T=\{\sigma_1,\sigma_2,...,\sigma_n\}$$

The diameter of each small region is denoted as

$$d_i=\max\{|x-y|\}\ (\forall x,y\in\sigma_i)$$

and

$$||T||=\max_{1\le i\le n}\{d_i\}$$

is called the fineness of T.

Definition 4 Let $f(x,y)$ be a function defined on D, and J be a real number. If for all $\varepsilon>0$, there exists a $\delta>0$ such that for any partition T of D and $(\xi_i,\eta_i)\in\sigma_i$, whenever $||T||<\delta$, we have

$$\left|\sum_{i=1}^{n} f(\xi_i,\eta_i)\sigma_i-J\right|<\varepsilon$$

then $f(x,y)$ is said to be integrable on D, and J is called the double integral of $f(x,y)$ on D, denoted as:

$$J = \iint_D f(x,y)\,d\sigma$$

corollary 2 Suppose $f(x)$ is Riemann integrable on $[a,b]$ and $g(x)$ is Riemann integrable on $[c,d]$, with $D = [a,b]\times[c,d]$. According to the definition of the double integral, we have:

$$\lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\cdot\frac{d-c}{n}\sum_{j=1}^{n} f\left(c + j\frac{d-c}{n}\right) = \int_a^b f(x)\,dx \int_c^d g(y)\,dy$$

Note: The second summand in the original text seems to be a mistake and should be related to $g(x)$ or another function, but for the sake of translation consistency, I kept it as $f(x)$ here. However, in practice, it should be corrected to match the context.

corollary 3 Suppose $f(x,y)$ is integrable on the region $D = [a,b]\times[c,d]$. According to the definition of the double integral, we have:

$$\lim_{n\to\infty} \frac{(b-a)(d-c)}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f\left(a + i\frac{b-a}{n}, c + j\frac{d-c}{n}\right) = \iint_D f(x,y)\,dx\,dy$$

e.g.13.2.1 Find:

$$\lim_{n\to\infty} \frac{\pi}{2n^4}\sum_{i=1}^{n}\sum_{j=1}^{n} i^2 \sin\frac{j\pi}{2n}$$

Solution:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\pi}{2n^4}\sum_{i=1}^{n}\sum_{j=1}^{n} i^2 \sin\frac{j\pi}{2n} \\
&= \frac{\pi}{2}\cdot\frac{1}{n^2}\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\sum_{j=1}^{n}\left(\sin\frac{\pi}{2}\cdot\frac{j}{n}\right) \\
&= \frac{\pi}{2}\int_0^1 x^2 dx \int_0^1 \left(\sin\frac{\pi}{2}y\right) dy \\
&= \boxed{\frac{1}{3}}
\end{aligned}$$

e.g.13.2.2 Find:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}$$

Solution:

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}\\&=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\frac{i}{n}+\frac{j}{n}}{(\frac{i}{n})^2+(\frac{j}{n})^2}\\&=\int_0^1\int_0^1\frac{x+y}{x^2+y^2}dxdy\end{aligned}$$

Using polar coordinates substitution, let $x=\rho\cos\theta, y=\rho\sin\theta$

$$\begin{aligned}&\int_0^1\int_0^1\frac{x+y}{x^2+y^2}dxdy\\&=2\int_0^{\frac{\pi}{4}}d\theta\int_0^{\frac{1}{\cos\theta}}\frac{\rho\cos\theta+\rho\sin\theta}{(\rho\cos\theta)^2+(\rho\sin\theta)^2}\rho d\rho\\&=2\int_0^{\frac{\pi}{4}}d\theta\int_0^{\frac{1}{\cos\theta}}(\sin\theta+\cos\theta)d\rho\\&=2\int_0^{\frac{\pi}{4}}\frac{(\sin\theta+\cos\theta)}{\cos\theta}d\theta\\&=2\left[1-\ln|\cos\theta|\right]_{\theta=0}^{\theta=\frac{\pi}{4}}\\&=\boxed{\frac{\pi}{2}+\ln 2}\end{aligned}$$

Triple and higher-order integrals are similar to double integrals and will not be elaborated further.

14 Gauss Integer Rounding Functions

Definition 1 $\lfloor x\rfloor$ is the largest integer not exceeding x, and $\lceil x\rceil$ is the smallest integer not less than x. $\lfloor x\rfloor$ and $\lceil x\rceil$ are collectively known as Gauss integer rounding functions.
$\lfloor x\rfloor$ is called the floor function, for example, $\lfloor 2.5\rfloor=2$, $\lfloor -1.5\rfloor=-2$.
$\lceil x\rceil$ is called the ceiling function, for example, $\lceil 2.5\rceil=3$, $\lceil -1.5\rceil=-1$.

corollary 1 If $\{x\}$ denotes the fractional part of x, then $\{x\}=x-\lfloor x\rfloor$.

e.g.14.1(Difficult): Solve for:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{2n}{k}\right\rfloor-2\left\lfloor\frac{n}{k}\right\rfloor\right)$$

Solution:

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{2n}{k}\right\rfloor-2\left\lfloor\frac{n}{k}\right\rfloor\right)\\&=\int_0^1\left(\left\lfloor\frac{2}{x}\right\rfloor-2\left\lfloor\frac{1}{x}\right\rfloor\right)dx\\&=\int_1^{+\infty}\frac{\lfloor 2t\rfloor-2\lfloor t\rfloor}{t^2}dt\quad (t=\frac{1}{x})\\&=\sum_{k=1}^{\infty}\int_{\frac{k}{2}}^{\frac{k+1}{2}}\frac{\lfloor 2t\rfloor-2\lfloor t\rfloor}{t^2}dt\end{aligned}$$

When $k\le 2t<k+1$, $\lfloor 2t\rfloor=k$.

If $k=2i$ (where $i=1,2,\ldots$) is even, then $i\le t<i+\frac{1}{2}$, and $\lfloor 2t\rfloor-2\lfloor t\rfloor=2i-2i=0$.

If $k=2i-1$ (where $i=2,3,\ldots$) is odd, then $i-\frac{1}{2}\le t<i$, and $\lfloor 2t\rfloor-2\lfloor t\rfloor=2i-1-2(i-1)=1$.

Therefore,

$$\begin{aligned}&\sum_{k=1}^{\infty}\int_{\frac{k}{2}}^{\frac{k+1}{2}}\frac{\lfloor 2t\rfloor-2\lfloor t\rfloor}{t^2}dt\\&=\sum_{i=2}^{\infty}\int_{i-\frac{1}{2}}^{i}\frac{1}{t^2}dt\\&=\sum_{i=2}^{\infty}\left(\frac{2}{2i-1}-\frac{2}{2i}\right)\\&=2\sum_{i=2}^{\infty}\left(\frac{1}{2i-1}-\frac{1}{2i}\right)\\&=2\sum_{i=1}^{\infty}\left(\frac{1}{2i-1}-\frac{1}{2i}\right)-1\end{aligned}$$

And since

$$\begin{aligned}&\sum_{i=1}^{\infty}\left(\frac{1}{2i-1}-\frac{1}{2i}\right)\\&=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{2i-1}-\frac{1}{2i}\right)\\&=\lim_{n\to\infty}\left[\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)\right]\\&=\lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)\right]\\&=\lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)\right]\\&=\ln 2\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{2n}{k}\right\rfloor-2\left\lfloor\frac{n}{k}\right\rfloor\right)=\boxed{2\ln 2-1}$$

e.g.14.2(Difficult): Solve for:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{n}{k}\right\}^2$$

(where {t} denotes the fractional part of t)
Solution:

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{\frac{n}{k}\right\}^2\\&=\int_0^1\left\{\frac{1}{x}\right\}^2dx\\&=\int_1^{+\infty}\frac{\{t\}^2}{t^2}dt\\&=\int_1^{+\infty}\frac{(t-\lfloor t\rfloor)^2}{t^2}dt\\&=\sum_{k=1}^{\infty}\int_k^{k+1}\frac{(t-k)^2}{t^2}dt\\&=\sum_{k=1}^{\infty}\int_k^{k+1}\left(1-\frac{2k}{t}+\frac{k^2}{t^2}\right)dt\\&=\sum_{k=1}^{\infty}\left(2-\frac{1}{k+1}-2k\ln\frac{k+1}{k}\right)\end{aligned}$$

Given that $\lim_{n\to\infty}\left(1+\frac{1}{2}+...+\frac{1}{n}-\ln n\right)=\gamma$, let $H_n=1+\frac{1}{2}+...+\frac{1}{n}$. Also, by Stirling's formula, $n!\sim\sqrt{2n\pi}\left(\frac{n}{e}\right)^n$. Then,

$$H_n=\ln n+\gamma+o(1)$$

$$\ln n!=\ln\sqrt{2\pi}+\left(n+\frac{1}{2}\right)\ln n-n+o(1)$$

Therefore,

$$\begin{aligned}&\sum_{k=1}^{n}\left(2-\frac{1}{k+1}-2k\ln\frac{k+1}{k}\right)\\&=2n-(H_{n+1}-1)-2\sum_{k=1}^{n}[(k+1)\ln(k+1)-k\ln k]+2\sum_{k=1}^{n}\ln(k+1)\\&=2n+1-H_{n+1}-2(n+1)\ln(n+1)+2\ln(n+1)!\\&=2n+1-(\ln(n+1)+\gamma+o(1))-2(n+1)\ln(n+1)\\&+2\left(\ln\sqrt{2\pi}+\left(n+\frac{3}{2}\right)\ln(n+1)-n-1+o(1)\right)\\&=-1-\gamma+\ln(2\pi)+o(1)\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{n}{k}\right)^2\\
&=\sum_{k=1}^{\infty}\left(2-\frac{1}{k+1}-2k\ln\frac{k+1}{k}\right)\\
&=\lim_{n\to\infty}\sum_{k=1}^{n}\left(2-\frac{1}{k+1}-2k\ln\frac{k+1}{k}\right)\\
&=\lim_{n\to\infty}\left(-1-\gamma+\ln(2\pi)+o(1)\right)\\
&=\boxed{-1-\gamma+\ln(2\pi)}
\end{aligned}$$

15 Approximation Method

The approximation method plays a significant role in solving limits or proving inequalities. The main idea of the approximation method is to ”transform” the expression to be proved, making it similar to the known and desired forms, thereby facilitating further manipulation.

For example, to prove $\lim_{n\to\infty} a_n = a$, it suffices to prove $\lim_{n\to\infty}(a_n - a) = 0$. While this sounds simple, in practice, a_n may be complex, while a is relatively simple. In such cases, simply moving a to the left side does not significantly help with solving the problem. Instead, we should transform a into a form similar to a_n to facilitate calculation.

[e.g.15.1]

Given $\lim_{n\to\infty} a_n = A$, prove that:

$$\lim_{n\to\infty}\frac{a_1+a_2+a_3+...+a_n}{n}=A$$

Proof: Starting from

$$A=\frac{nA}{n}=\frac{A+A+...+A}{n}$$

we have

$$\begin{aligned}
&\left|\frac{a_1+a_2+a_3+...+a_n}{n}-A\right|\\
&=\left|\frac{a_1-A+a_2-A+...+a_n-A}{n}\right|\\
&\le\frac{|a_1-A|+|a_2-A|+...+|a_n-A|}{n}
\end{aligned}$$

Since $\lim_{n\to\infty} a_n = A$, for any $\varepsilon > 0$, there exists $N_0 > 0$ such that for all $n > N_0$, $|a_n - A| < \varepsilon$. Therefore,

$$\begin{aligned}
&\frac{|a_1-A|+|a_2-A|+...+|a_n-A|}{n}\\
&=\frac{|a_1-A|+|a_2-A|+...+|a_{N_0}-A|}{n}\\
&\quad+\frac{|a_{N_0+1}-A|+|a_{N_0+2}-A|+...+|a_n-A|}{n}\\
&<\frac{M}{n}+\frac{n-N_0}{n}\varepsilon\\
&<2\varepsilon
\end{aligned}$$

where M is a constant representing the sum of the absolute differences $|a_i - A|$ for $i = 1, 2, ..., N_0$.
Hence,

$$\lim_{n\to\infty} \frac{a_1 + a_2 + a_3 + ... + a_n}{n} = A$$

[e.g.15.2]

Given $f(x) \sim x$ as $x \to 0$, and $x_n = \sum_{i=1}^{n} f\left(\frac{2i-1}{n^2}a\right)$, prove that:

$$\lim_{n\to\infty} x_n = a$$

Proof: In fact, if $f(x) = x$, then

$$\sum_{i=1}^{n} f\left(\frac{2i-1}{n^2}a\right) = \sum_{i=1}^{n} \frac{2i-1}{n^2}a = a$$

Therefore, we use $\sum_{i=1}^{n} \frac{2i-1}{n^2}a$ to approximate a.
We only need to prove that $|x_n - a| < \varepsilon$, i.e.,

$$\begin{aligned} &|x_n - a| \\ &= \left|\sum_{i=1}^{n} f\left(\frac{2i-1}{n^2}a\right) - \sum_{i=1}^{n} \frac{2i-1}{n^2}a\right| \\ &\leq \sum_{i=1}^{n} \left|f\left(\frac{2i-1}{n^2}a\right) - \frac{2i-1}{n^2}a\right| \\ &< \varepsilon \end{aligned}$$

Since $\varepsilon = \sum_{i=1}^{n} \frac{2i-1}{n^2}\varepsilon$, we only need

$$\left|f\left(\frac{2i-1}{n^2}a\right) - \frac{2i-1}{n^2}a\right| < \frac{2i-1}{n^2}\varepsilon$$

In fact, since $f(x) \sim x$ as $x \to 0$, for any $\varepsilon > 0$, there exists $N > 0$ such that when $n > N$ and δ is small enough, if $0 < \left|x - \frac{2i-1}{n^2}a\right| < \delta$, then

$$\left|\frac{f\left(\frac{2i-1}{n^2}a\right)}{\frac{2i-1}{n^2}a} - 1\right| < \frac{\varepsilon}{a}$$

Thus,

$$\left|f\left(\frac{2i-1}{n^2}a\right) - \frac{2i-1}{n^2}a\right| < \frac{2i-1}{n^2}\varepsilon$$

This completes the proof. □

Note: Since $x \sim \sin x \sim \tan x \sim \arcsin x \sim \arctan x \sim e^x - 1 \sim \ln(1+x)$ as $x \to 0$, replacing x with other functions also holds.

[e.g.15.3]

Given $\lim_{n\to\infty} a_n = a$, prove that:

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} C_n^k a_k = a$$

Proof: Since

$$1 = \frac{(1+1)^n}{2^n} = \frac{1}{2^n} \sum_{k=0}^{n} C_n^k$$

we have

$$a = \frac{1}{2^n} \sum_{k=0}^{n} C_n^k a$$

Then,

$$\begin{aligned}&\left|\frac{1}{2^n}\sum_{k=0}^{n}C_n^k a_k - a\right|\\&=\left|\frac{1}{2^n}\sum_{k=0}^{n}C_n^k a_k - \frac{1}{2^n}\sum_{k=0}^{n}C_n^k a\right|\\&\le \frac{1}{2^n}\sum_{k=0}^{n}C_n^k|a_k - a|\end{aligned}$$

Since $\lim_{n\to\infty} a_n = a$, then $|a_n| < M$, and

$$\begin{aligned}&\frac{1}{2^n}\sum_{k=0}^{n}C_n^k|a_k - a|\\&=\frac{1}{2^n}\sum_{k=0}^{N_0}C_n^k|a_k - a| + \frac{1}{2^n}\sum_{k=N_0+1}^{n}C_n^k|a_k - a|\\&<\frac{M(1+n+n^2+...+n^{N_0})}{2^n} + \varepsilon\sum_{k=N_0+1}^{n}\frac{C_n^k}{2^n}\\&< 2\varepsilon\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=0}^{n}C_n^k a_k = a$$

[e.g.15.4] Find the limit:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^2+k} - \frac{n}{3}$$

Solution: We can attempt to approximate $\frac{n}{3}$. Although

$$\sum_{k=1}^{n}\frac{k^2}{n^2+k} \sim \frac{n}{3} \quad \text{as} \quad n\to\infty$$

this does not directly help with solving the problem. However, if we notice that

$$\sum_{k=1}^{n}\frac{k^2}{n^2} = \frac{n(n+1)(2n+1)}{6n^2} = \frac{n}{3} + \frac{1}{2} + \frac{1}{6n}$$

then

$$\begin{aligned}&\sum_{k=1}^{n}\frac{k^2}{n^2+k} - \frac{n}{3}\\&=\sum_{k=1}^{n}\frac{k^2}{n^2+k} - \left(\sum_{k=1}^{n}\frac{k^2}{n^2} - \frac{1}{2} - \frac{1}{6n}\right)\\&=\sum_{k=1}^{n}\left(\frac{k^2}{n^2+k} - \frac{k^2}{n^2}\right) + \frac{1}{2} + \frac{1}{6n}\end{aligned}$$

Since

$$\sum_{k=1}^{n}\left(\frac{k^2}{n^2+k} - \frac{k^2}{n^2}\right) = -\sum_{k=1}^{n}\frac{k^3}{n^2(n^2+k)}$$

and

$$\sum_{k=1}^{n} \frac{k^3}{n^2(n^2+n)} < \sum_{k=1}^{n} \frac{k^3}{n^2(n^2+k)} < \sum_{k=1}^{n} \frac{k^3}{n^2(n^2+1)}$$

also,

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

therefore

$$\sum_{k=1}^{n} \frac{k^3}{n^2(n^2+n)} = \frac{1}{4} \cdot \frac{n+1}{n}$$
$$\sum_{k=1}^{n} \frac{k^3}{n^2(n^2+1)} = \frac{1}{4} \cdot \frac{(n+1)^2}{n^2+1}$$

Thus,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3}{n^2(n^2+k)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3}{n^2(n^2+n)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3}{n^2(n^2+1)} = \frac{1}{4}$$

So,

$$\begin{aligned} &\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k^2}{n^2+k} - \frac{k^2}{n^2} \right) + \frac{1}{2} + \frac{1}{6n} \\ &= -\frac{1}{4} + \frac{1}{2} \\ &= \boxed{\frac{1}{4}} \end{aligned}$$

[e.g.15.5] If $f(x) \in C[0,1]$, prove that:

$$\lim_{n\to\infty} \int_0^1 \frac{n}{1+n^2x^2} f(x)dx = \frac{\pi}{2} f(0)$$

Proof: We will prove a stronger statement: It suffices to prove that

$$\lim_{h\to 0^+} \int_0^1 \frac{h}{x^2+h^2} f(x)dx = \frac{\pi}{2} f(0)$$

Then, by Heine's theorem (the principle of the limit under the integral sign), taking the sequence $x_n = \frac{1}{n} \to 0^+$ (as $n \to +\infty$), we have:

$$\lim_{n\to\infty} \int_0^1 \frac{\frac{1}{n}}{x^2+(\frac{1}{n})^2} f(x)dx = \lim_{n\to\infty} \int_0^1 \frac{n}{1+n^2x^2} f(x)dx = \frac{\pi}{2} f(0)$$

Next, we prove that

$$\lim_{h\to 0^+} \int_0^1 \frac{h}{x^2+h^2} f(x)dx = \frac{\pi}{2} f(0)$$

Proof: In fact, if $f(x) = 1$, then

$$\lim_{h\to 0^+} \int_0^1 \frac{h}{x^2+h^2} dx = \frac{\pi}{2}$$

Therefore, it suffices to prove that:

$$\lim_{h\to 0^+} \int_0^1 \frac{h}{x^2+h^2} [f(x) - f(0)]dx = 0$$

Since $f(x)$ is continuous on $[0,1]$, we have:

$$\begin{aligned}&\forall\varepsilon>0,\exists 0<\delta_1<1,\forall 0<x<\delta_1 \text{ s.t. } |f(x)-f(0)|<\frac{\varepsilon}{\pi}\\&\exists M>0,\forall x\in[0,1],\ \text{s.t. } |f(x)|\le M\end{aligned}$$

Then,

$$\begin{aligned}&\left|\int_0^1\frac{h}{x^2+h^2}[f(x)-f(0)]dx\right|\\&\le\left(\int_0^{\delta_1}+\int_{\delta_1}^1\right)\frac{h}{x^2+h^2}|f(x)-f(0)|dx\\&=I_1+I_2\end{aligned}$$

For I_1:

$$\begin{aligned}I_1&=\int_0^{\delta_1}\frac{h}{x^2+h^2}|f(x)-f(0)|dx\\&\le\frac{\varepsilon}{\pi}\int_0^{\delta_1}\frac{h}{x^2+h^2}dx\\&=\frac{\varepsilon}{\pi}(\arctan\frac{\delta_1}{h}-0)\end{aligned}$$

As $h\to 0^+$, $\frac{\delta_1}{h}\to+\infty$, so

$$\frac{\varepsilon}{\pi}\arctan\frac{\delta_1}{h}\le\frac{\varepsilon}{2}$$

For I_2:

$$\begin{aligned}I_2&=\int_{\delta_1}^1\frac{h}{x^2+h^2}|f(x)-f(0)|dx\\&\le M\int_{\delta_1}^1\frac{h}{x^2+h^2}dx\\&=M(\arctan\frac{1}{h}-\arctan\frac{\delta_1}{h})\end{aligned}$$

Since

$$\lim_{h\to 0^+}M(\arctan\frac{1}{h}-\arctan\frac{\delta_1}{h})=M(\frac{\pi}{2}-\frac{\pi}{2})=0$$

There exists $\delta_2>0$, $\forall 0<h<\delta_2$, s.t.

$$M(\arctan\frac{1}{h}-\arctan\frac{\delta_1}{h})\le\frac{\varepsilon}{2}$$

Therefore, $\forall\varepsilon>0$, $\exists\delta=\min\{\delta_1,\delta_2\}$, $\forall 0<h<\delta$, s.t.

$$\left|\int_0^1\frac{h}{x^2+h^2}[f(x)-f(0)]dx\right|\le\varepsilon$$

Thus,

$$\lim_{h\to 0^+}\int_0^1\frac{h}{x^2+h^2}f(x)dx=\frac{\pi}{2}f(0)$$

[e.g.15.6] For all $b > 0$, $f(x) \in R[0, b]$, and $\lim_{x\to+\infty} f(x) = a$, prove that:

$$\lim_{t\to0^+} t \int_0^{+\infty} e^{-tx} f(x)dx = a$$

Proof: Since when $t \neq 0$, we have

$$t \int_0^{+\infty} e^{-tx} a dx = a$$

It suffices to prove that

$$\lim_{t\to0^+} t \int_0^{+\infty} e^{-tx} [f(x) - a]dx = 0$$

Since $\lim_{x\to+\infty} f(x) = a$, then $\forall \varepsilon > 0$, $\exists A > 0$, $\forall x > A$, s.t. $|f(x) - a| < \varepsilon$.
And since $\forall b > 0$, $f(x) \in R[0, b]$, $f(x)$ is bounded on $[0, A]$, i.e.,

$$|f(x)| \leq M', \quad |f(x) - a| \leq |f(x)| + |a| \leq M$$

Therefore,

$$\begin{aligned}
&\left| t \int_0^{+\infty} e^{-tx} [f(x) - a] \right| \\
&\leq t \int_0^{+\infty} e^{-tx} |f(x) - a| \\
&= t \int_0^{A} e^{-tx} |f(x) - a| + t \int_A^{+\infty} e^{-tx} |f(x) - a| \\
&\leq Mt \int_0^{A} e^{-tx} dx + \varepsilon \cdot t \int_A^{+\infty} e^{-tx} dx \\
&= M(1 - e^{-At}) + \varepsilon e^{-At} \to 0 \quad (t \to 0^+)
\end{aligned}$$

Thus,

$$\lim_{t\to0^+} t \int_0^{+\infty} e^{-tx} f(x)dx = a$$

Q.E.D.

16 Necessary Condition for Convergence of Series

Theorem 1 A necessary and sufficient condition for the convergence of the series $\sum a_n$ is that

$$\lim_{n\to\infty} a_n = 0.$$

Therefore, if the series $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Below are some commonly used methods to determine the convergence or divergence of series:

Theorem 2 (Cauchy's Criterion for Convergence of Series) Let $\sum a_n$ be a series. A necessary and sufficient condition for its convergence is: $\forall \varepsilon > 0, \exists N > 0$ such that for all $m > N, p \in \mathbb{N}$, we have

$$|a_{m+1} + a_{m+2} + ... + a_{m+p}| < \varepsilon.$$

Theorem 3 (Comparison Test) For a positive series $\sum a_n$, if there exists a convergent positive series $\sum b_n$ and an N such that $a_n \leq b_n$ for all $n > N$, then the series $\sum a_n$ converges.
If there exists a divergent positive series $\sum c_n$ and an N such that $a_n \geq c_n$ for all $n > N$, then the series $\sum a_n$ diverges.

corollary 1 Let $\sum a_n$ and $\sum b_n$ be two positive series. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l,$$

(1) If $0 < l < +\infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge together;
(2) If $l = 0$, then if $\sum b_n$ converges, $\sum a_n$ also converges;
(3) If $l = +\infty$, then if $\sum b_n$ diverges, $\sum a_n$ also diverges.

Theorem 4 (Ratio Test) Let $\sum a_n$ be a positive series. If there exists an N and a constant q such that when $n > N$,
(1)

$$\frac{a_{n+1}}{a_n} \leq q,$$

then the series $\sum a_n$ converges;
(2)

$$\frac{a_{n+1}}{a_n} \geq 1,$$

then the series $\sum a_n$ diverges.

corollary 2 Let $\sum a_n$ be a positive series, and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q.$$

(1) If $q < 1$, then the series $\sum a_n$ converges;
(2) If $q > 1$ or $q = +\infty$, then the series $\sum a_n$ diverges.

corollary 3 Let $\sum a_n$ be a positive series.
(1) If

$$\overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} = q < 1,$$

then the series $\sum a_n$ converges;
(2) If

$$\underline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} = q > 1,$$

then the series $\sum a_n$ diverges.

Theorem 5 (Root Test) Let $\sum a_n$ be a positive series. If there exists an N and a constant l such that when $n > N$,
(1)

$$\sqrt[n]{a_n} \leq l < 1,$$

then the series $\sum a_n$ converges;
(2)

$$\sqrt[n]{a_n} \geq 1,$$

then the series $\sum a_n$ diverges.

corollary 4 Let $\sum a_n$ be a positive series, and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = l.$$

(1) If $l < 1$, then the series $\sum a_n$ converges;
(2) If $l > 1$ or $l = +\infty$, then the series $\sum a_n$ diverges.

corollary 5 Let $\sum a_n$ be a positive series.
(1) If

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = l < 1,$$

then the series $\sum a_n$ converges;
(2) If

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = l > 1,$$

then the series $\sum a_n$ diverges.

Theorem 6
(Integral Test) Suppose $f(x)$ is a nonnegative, monotonically decreasing function on $[1, +\infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral $\int_1^{+\infty} f(x)\,dx$ converges.

corollary 6
Suppose $f(x)$ is a nonnegative, monotonically decreasing function on $[N_0, +\infty)$. Then the series $\sum_{n=N_0}^{\infty} f(n)$ converges if and only if the improper integral $\int_{N_0}^{+\infty} f(x)\,dx$ converges.

Theorem 7
(Raabe's Test) Suppose $\sum a_n$ is a series of positive terms. If there exists $N > 0$ and a constant r such that for $n > N$:
(1) $n\left(1 - \frac{a_{n+1}}{a_n}\right) > 1$, then $\sum a_n$ converges;
(2) $n\left(1 - \frac{a_{n+1}}{a_n}\right) \le 1$, then $\sum a_n$ diverges.

corollary 7
Suppose $\sum a_n$ is a series of positive terms, and
$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = r$.
(1) If $r > 1$, then $\sum a_n$ converges;
(2) If $r < 1$, then $\sum a_n$ diverges.

For general series, the following are commonly used tests:

Theorem 8
(Leibniz's Test) If the alternating series $\sum(-1)^n a_n$ (where $a_n > 0$) satisfies:
(1) a_n is monotonically decreasing;
(2) $\lim_{n\to\infty} a_n = 0$
then the series $\sum(-1)^n a_n$ converges.

Theorem 9
A series that converges absolutely also converges.

Theorem 10
(Abel's Test) If $\{a_n\}$ is monotonically bounded and $\sum b_n$ converges, then $\sum a_n b_n$ converges.

Theorem 11
(Dirichlet's Test) If $\{a_n\}$ is monotonically decreasing to 0 and the partial sums of $\sum b_n$ are bounded, then $\sum a_n b_n$ converges.

e.g.16.1 Proof:

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Proof: Consider the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$. By the ratio test,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \\
&= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} \\
&= \frac{1}{e} < 1
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{n!}{n^n}=0$$

17 Toeplitz Theorem and Abel's Transformation

In the process of finding limits, we sometimes encounter problems involving the sum of the products of two sequences, denoted as $\lim_{n\to\infty}\sum a_n b_n$. Sometimes we can attempt to use the definition of the definite integral or the squeeze theorem. Sometimes, we need to transform the summation formula to solve for the limit. The Toeplitz theorem can solve a special class of limit problems involving the sum of the products of sequences. Abel's transformation (also known as summation by parts) can reduce the "degree" of the summation formula, thereby simplifying it into a more straightforward form and making it easier to find the limit.

17.1 Toeplitz Theorem

Theorem 1 Let $y_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \sum_{j=1}^{n} a_{nj}x_j$. If the following conditions are satisfied:
(1) $a_{nj} \geq 0, j = 1, 2, \cdots, n$;
(2) $\sum_{j=1}^{n} a_{nj} = 1$;
(3) $\forall j, \lim_{n\to\infty} a_{nj} = 0$.
If $\lim_{n\to\infty} x_n = l$, then $\lim_{n\to\infty}\sum_{j=1}^{n} a_{nj}x_j = l$.

Theorem 2 Let $y_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \sum_{j=1}^{n} a_{nj}x_j$. If the following conditions are satisfied:
(1) $\exists k > 0$ such that $\forall n \in N$, $|a_{n1}| + |a_{n2}| + \cdots + |a_{nn}| \leq k$;
(2) $\forall j, \lim_{n\to\infty} a_{nj} = 0$.
If $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty}\sum_{j=1}^{n} a_{nj}x_j = 0$.

Theorem 3 Let $y_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \sum_{j=1}^{n} a_{nj}x_j$. If the following conditions are satisfied:
(1) $\exists k > 0$ such that $\forall n \in N$, $|a_{n1}| + |a_{n2}| + \cdots + |a_{nn}| \leq k$;
(2) $\forall j, \lim_{n\to\infty} a_{nj} = 0$;
(3) $\lim_{n\to\infty}\sum_{j=1}^{n} a_{nj} = 1$.
If $\lim_{n\to\infty} x_n = l$, then $\lim_{n\to\infty}\sum_{j=1}^{n} a_{nj}x_j = l$.

[e.g.17.1.1] $\lim_{n\to\infty} x_n = a$,prove that:

$$\lim_{n\to\infty}\frac{a_1 + a_2 + \cdots + a_n}{n} = a$$

Proof: Take $t_{nj} = \frac{1}{n}$, which satisfies the conditions of Theorem 1, thus:

$$\lim_{n\to\infty}\frac{a_1 + a_2 + \cdots + a_n}{n} = \lim_{n\to\infty} x_n = a$$

[e.g.17.1.2] If $\lim_{n\to\infty} x_n = a, \lim_{n\to\infty} y_n = b$,prove that:

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\frac{x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1}{n} = ab$$

Proof:

(1) If $a = 0$, take $t_{nj} = \frac{y_i}{n}$, which satisfies the conditions of Theorem 2, thus $\lim_{n\to\infty} z_n = 0$
(2) If $a \neq 0$, since

$$z_n = \frac{(x_1 - a)y_n + (x_2 - a)y_{n-1} + \cdots + (x_n - a)y_1}{n} + a\frac{y_1 + y_2 + \cdots + y_n}{n}$$

where the limit of the first term is 0 by (1), and the limit of the second term is ab by **[e.g.17.1.1]** □

[e.g.17.1.3] Let $p_i > 0$. If $\lim_{n\to\infty} \frac{p_n}{p_0+p_1+\cdots+p_n} = 0, \lim_{n\to\infty} S_n = s$,prove that:

$$\lim_{n\to\infty} \frac{S_0 p_n + S_1 p_{n-1} + \cdots + S_n p_0}{p_0 + p_1 + \cdots + p_n} = s$$

Proof: Let $a_{nj} = \frac{p_{n-j}}{p_0+p_1+\cdots+p_n}$, then $a_{nj} > 0, \sum_{j=0}^{n} a_{nj} = 1$, and for all j,

$$0 < a_{nj} = \frac{p_{n-j}}{p_0 + p_1 + \cdots + p_n} < \frac{p_{n-j}}{p_0 + p_1 + \cdots + p_{n-1}} \to 0 \quad (n \to \infty)$$

Thus, $\lim_{n\to\infty} a_{nj} = 0$
Hence, by Theorem 1,

$$\lim_{n\to\infty} \frac{S_0 p_n + S_1 p_{n-1} + \cdots + S_n p_0}{p_0 + p_1 + \cdots + p_n} = s$$

17.2 Abel's Transformation

Theorem 4 Let $B_k = \sum_{i=1}^{k} b_i$ with $B_0 = 0$, then

$$\sum_{k=1}^{n} a_k b_k = a_n B_n - \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k$$

[e.g. 17.2.1 (Difficult)] Prove that:

$$\lim_{x\to 0} \sum_{i=1}^{\infty} \frac{\cos ix}{i} = +\infty$$

Proof: First, consider $S_n = \sum_{i=1}^{n} \cos ix = \sum_{i=1}^{n} \frac{\cos ix \cdot \sin \frac{x}{2}}{\sin \frac{x}{2}}$. Using the identity

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

we have

$$S_n = \sum_{i=1}^{n} \frac{\sin(i+\frac{1}{2})x - \sin(i-\frac{1}{2})x}{2\sin\frac{x}{2}} = \frac{\sin(n+\frac{1}{2})x - \sin\frac{x}{2}}{2\sin\frac{x}{2}}$$

Thus, S_n is bounded. By Dirichlet's test, the infinite series converges.
Since $\cos(-x) = \cos x$, we may assume $x > 0$ without loss of generality.

Applying Abel's transformation, we have

$$\begin{aligned}
&\sum_{i=1}^{\infty}\frac{\cos ix}{i}\\
&=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{S_i-S_{i-1}}{i}\\
&=\lim_{n\to\infty}\left(\frac{S_n}{n}-\sum_{i=1}^{n-1}S_i\left(\frac{1}{i+1}-\frac{1}{i}\right)\right)\\
&=\lim_{n\to\infty}\sum_{i=1}^{n-1}\frac{S_i}{i(i+1)}\\
&=\sum_{i=1}^{\infty}\frac{S_i}{i(i+1)}
\end{aligned}$$

For $\forall 0<x<\frac{1}{2}$, choose n_0 such that

$$x(n_0-\frac{1}{2})\leq\frac{\pi}{2}<x(n_0+\frac{1}{2})$$

and use the inequality $\sin x>\frac{2}{\pi}x$ for $0<x<\frac{\pi}{2}$.
We obtain

$$\begin{aligned}
\sum_{i=1}^{\infty}\frac{S_i}{i(i+1)} &= \sum_{i=1}^{\infty}\frac{1}{2\sin\frac{x}{2}}\frac{\sin(i+\frac{1}{2})x}{i(i+1)}-\frac{1}{2}\frac{1}{i(i+1)}\\
&\geqslant -\frac{1}{2}+\frac{1}{2\sin\frac{x}{2}}\sum_{i=1}^{\infty}\frac{\sin(i+\frac{1}{2})x}{i(i+1)}\\
&\geqslant -\frac{1}{2}+\frac{1}{x}\sum_{i=1}^{\infty}\frac{\sin(i+\frac{1}{2})x}{i(i+1)}\\
&= -\frac{1}{2}+\frac{1}{x}\left[\sum_{i=1}^{n_0-1}\frac{\sin(i+\frac{1}{2})x}{i(i+1)}+\sum_{i=n_0}^{\infty}\frac{\sin(i+\frac{1}{2})x}{i(i+1)}\right]\\
&\geqslant -\frac{1}{2}+\frac{1}{x}\left[\sum_{i=1}^{n_0-1}\frac{\sin(i+\frac{1}{2})x}{i(i+1)}-\sum_{i=n_0}^{\infty}\frac{1}{i(i+1)}\right]\\
&\geqslant -\frac{1}{2}+\frac{1}{x}\frac{2}{\pi}\sum_{i=1}^{n_0-1}\frac{(i+\frac{1}{2})x}{i(i+1)}-\frac{1}{xn_0}\\
&\geqslant -\frac{1}{2}+\frac{2}{\pi}\sum_{i=1}^{n_0-1}\frac{1}{i+1}-\frac{1}{xn_0}\\
&\geqslant -\frac{1}{2}+\frac{2}{\pi}(\ln n_0-1)-\frac{4}{2\pi-1}
\end{aligned}$$

As $x\to 0^+$, $n_0\to\infty$, thus $\lim_{x\to 0}\sum_{i=1}^{\infty}\frac{\cos ix}{i}=+\infty$. □

[e.g. 17.2.2] Suppose $\{a_n\}$ is strictly monotonically increasing and tends to $+\infty$, with $a_n>0$, and $\sum_{k=1}^{\infty}b_k$ converges to B. Prove that:

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}a_kb_k}{a_n}=0$$

Proof: Let $B_n = \sum_{i=1}^{n} b_i$, then $\lim_{n\to\infty} B_n = B$. By Abel's transformation, we have:

$$\begin{aligned}&\sum_{k=1}^{n} a_k b_k\\&= a_n B_n - \sum_{i=1}^{n-1}(a_{i+1}-a_i)B_i\\&= a_n B_n - \sum_{i=1}^{n-1}(a_{i+1}-a_i)B_i + \sum_{i=1}^{n-1}(a_{i+1}-a_i)B - (a_n - a_1)B\\&= a_1 B + a_n(B_n - B) - \sum_{i=1}^{n-1}(a_{i+1}-a_i)(B_i - B)\end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{a_1 B + a_n(B_n - B)}{a_n} = 0$$

it suffices to prove

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1}(a_{i+1}-a_i)(B_i - B)}{a_n} = 0$$

And

$$\begin{aligned}&\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1}(a_{i+1}-a_i)(B_i - B)}{a_n}\\&= \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(a_{i+1}-a_i)(B_i - B) - \sum_{i=1}^{n-1}(a_{i+1}-a_i)(B_i - B)}{a_{n+1}-a_n}\\&= \lim_{n\to\infty}(B_n - B)\\&= 0\end{aligned}$$

Proof completed. □

[e.g.17.2.3(difficult)] For $\forall m \geq 2$ and $m \in N$, prove that:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^m}{n^m + i \cdot n^{m-2}} - \frac{n}{m+1} = \frac{1}{2} - \frac{1}{m+2}$$

Proof: Before the formal proof, we need to know that

$$\sum_{i=1}^{n} \frac{i^m}{n^m} = \frac{n}{m+1} + \frac{1}{2} + o(1)(n \to \infty)$$

Now we prove this conclusion: It is easy to know that $\sum_{i=1}^{n} i^m = a_{m+1}n^{m+1} + a_m n^m + \cdots + a_1 n + a_0$ Therefore,

$$\sum_{i=1}^{n} \frac{i^m}{n^m} = a_{m+1}n + a_m + o(1)(n \to \infty)$$

Thus,

$$a_{m+1} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i^m}{n^m} = \int_0^1 x^m dx = \frac{1}{m+1}$$

Then,

$$a_m = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{i^m}{n^m} - \frac{n}{m+1}\right)$$

According to the conclusion of Example **[e.g.13.1.3]** in **??**, we have

$$a_m = \lim_{n\to\infty} n\left(\frac{1}{n}\sum_{i=1}^{n}\frac{i^m}{n^m} - \int_0^1 x^m dx\right) = \frac{1}{2}$$

Therefore,

$$\sum_{i=1}^{n}\frac{i^m}{n^m} = \frac{n}{m+1} + \frac{1}{2} + o(1)(n\to\infty)$$

Next, we use two methods to prove this problem:

(Method 1) Abel's Transformation:

Let $S_n = \sum_{i=1}^{n} i^m$ and apply Abel's transformation to obtain:

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{i^m}{n^m + i\cdot n^{m-2}}\\
&= \frac{S_n}{n^m + n^{m-1}} - \sum_{i=1}^{n-1}\left(\frac{1}{n^m + (i+1)n^{m-2}} - \frac{1}{n^m + i\cdot n^{m-2}}\right)S_i\\
&= \frac{S_n}{n^m + n^{m-1}} + \sum_{i=1}^{n-1}\frac{S_i}{n^{m-2}}\frac{1}{(n^2+i+1)(n^2+i)}
\end{aligned}$$

Consider

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^m}{n^m + i\cdot n^{m-2}} - \frac{n}{m+1}\\
&= \lim_{n\to\infty}\left(\frac{S_n}{n^m + n^{m-1}} - \frac{n}{m+1}\right) + \lim_{n\to\infty}\left(\sum_{i=1}^{n-1}\frac{S_i}{n^{m-2}}\frac{1}{(n^2+i+1)(n^2+i)}\right)
\end{aligned}$$

For the first term:

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{S_n}{n^m + n^{m-1}} - \frac{n}{m+1}\right)\\
&= \lim_{n\to\infty}\left(\frac{S_n}{n^m}\cdot\frac{n}{n+1} - \frac{n}{m+1}\right)\\
&= \lim_{n\to\infty}\left(\frac{n}{m+1} + \frac{1}{2} + o(1)\right)\frac{n}{n+1} - \frac{n}{m+1}\\
&= \lim_{n\to\infty}\frac{1}{2} - \frac{n}{(n+1)(m+1)}\\
&= \frac{1}{2} - \frac{1}{m+1}
\end{aligned}$$

For the second term, first consider

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n-1}\frac{S_i}{n^{m-2}}\frac{1}{(n^2+0)(n^2+0)}\right)$$

Then,

$$\begin{aligned}
&\lim_{n\to\infty}\left(\sum_{i=1}^{n-1}\frac{S_i}{n^{m-2}}\frac{1}{n^4}\right)\\
&=\lim_{n\to\infty}\left(\sum_{i=1}^{n-1}\frac{\sum_{j=1}^{i}j^m}{n^m}\frac{1}{n^2}\right)\\
&=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{i}(\frac{j}{n})^m-\lim_{n\to\infty}\frac{1}{n^2}\sum_{j=1}^{n}(\frac{j}{n})^m\\
&=\int_0^1 dx\int_0^x y^m dy\\
&=\frac{1}{(m+1)(m+2)}\\
&=\frac{1}{m+1}-\frac{1}{m+2}
\end{aligned}$$

By the Cauchy-Schwarz inequality $\sum a_nb_n \le \sqrt{(\sum a_n^2)(\sum b_n^2)}(a_n, b_n > 0)$, and $\sum a_n^2 \le (\sum a_n)^2(a_n > 0)$, therefore,

$$\sum a_nb_n \le \sum a_n\cdot\sqrt{(\sum b_n^2)}$$

Thus,

$$\begin{aligned}
&\left|\sum_{i=1}^{n-1}\frac{S_i}{n^{m-2}}\frac{1}{(n^2+i+1)(n^2+i)}-\sum_{i=1}^{n-1}\frac{S_i}{n^{m-2}}\frac{1}{n^4}\right|\\
&=\left|\sum_{i=1}^{n-1}\frac{S_i}{n^{m+2}}\frac{2n^2i+n^2+i^2+i}{(n^2+i)(n^2+i+1)}\right|\\
&\le\left|\sum_{i=1}^{n-1}\frac{S_i}{n^{m+2}}\right|\cdot\sqrt{\sum_{i=1}^{n-1}\left|\left[\frac{2n^2i+n^2+i^2+i}{(n^2+i)(n^2+i+1)}\right]^2\right|}\\
&\le\left|\sum_{i=1}^{n-1}\frac{S_i}{n^{m+2}}\right|\cdot\sqrt{\sum_{i=1}^{n-1}\left(\frac{Mn^3}{n^4}\right)^2}\\
&=\left|\sum_{i=1}^{n-1}\frac{S_i}{n^{m+2}}\right|\cdot\sqrt{\sum_{i=1}^{n-1}\left(\frac{M^2}{n^2}\right)}\\
&\le\left|\sum_{i=1}^{n-1}\frac{S_i}{n^{m+2}}\right|\cdot\sqrt{\left(\frac{M^2}{n}\right)}\\
&=\left|\sum_{i=1}^{n-1}\frac{S_i}{n^{m+2}}\right|\cdot\frac{M}{\sqrt{n}}\to 0(n\to\infty)
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n-1}\frac{S_i}{n^{m-2}}\frac{1}{(n^2+i+1)(n^2+i)}\right)=\frac{1}{m+1}-\frac{1}{m+2}$$

Thus,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^m}{n^m+i\cdot n^{m-2}}-\frac{n}{m+1}=\frac{1}{2}-\frac{1}{m+1}+\frac{1}{m+1}-\frac{1}{m+2}=\frac{1}{2}-\frac{1}{m+2}$$

Proof completed.

(Method 2) Approximation Method

Consider

$$\sum_{i=1}^{n}\frac{i^m}{n^m}=\frac{n}{m+1}+\frac{1}{2}+o(1)\quad (n\to\infty)$$

Therefore,

$$\frac{n}{m+1}=\sum_{i=1}^{n}\frac{i^m}{n^m}-\frac{1}{2}+o(1)$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{i^m}{n^m+i\cdot n^{m-2}}-\frac{n}{m+1}\\
&=\sum_{i=1}^{n}\frac{i^m}{n^m+i\cdot n^{m-2}}-\sum_{i=1}^{n}\frac{i^m}{n^m}+\frac{1}{2}+o(1)\\
&=\sum_{i=1}^{n}\left(\frac{i^m}{n^m+i\cdot n^{m-2}}-\frac{i^m}{n^m}\right)+\frac{1}{2}+o(1)\\
&=\frac{1}{2}-\sum_{i=1}^{n}\frac{i^{m+1}}{n^2(n^m+i\cdot n^{m-2})}+o(1)
\end{aligned}$$

Consider

$$\sum_{i=1}^{n}\frac{i^{m+1}}{n^2(n^m+n^{m-1})}<\sum_{i=1}^{n}\frac{i^{m+1}}{n^2(n^m+i\cdot n^{m-2})}<\sum_{i=1}^{n}\frac{i^{m+1}}{n^{m+2}}$$

Since

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{m+1}}{n^{m+2}}=\int_0^1 x^{m+1}dx=\frac{1}{m+2}$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{m+1}}{n^2(n^m+n^{m-1})}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{m+1}}{n^{m+2}}\cdot\lim_{n\to\infty}\frac{n}{1+n}=\frac{1}{m+2}$$

Therefore,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{m+1}}{n^2(n^m+i\cdot n^{m-2})}=\frac{1}{m+2}$$

Thus,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^m}{n^m+i\cdot n^{m-2}}-\frac{n}{m+1}=\frac{1}{2}-\frac{1}{m+2}$$

Proof completed.

18 Superior and Inferior Limits

Theorem 1 If every neighborhood of a number a contains infinitely many terms of the sequence $\{x_n\}$, then a is called a cluster point of the sequence $\{x_n\}$.

corollary 1 If a sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with $\lim_{k\to\infty} x_{n_k} = a$, then a is a cluster point of the sequence $\{x_n\}$.

Theorem 2 A bounded sequence $\{x_n\}$ has at least one cluster point, and there exist maximum and minimum cluster points.

Definition 1 The maximum cluster point of a bounded sequence $\{x_n\}$ is denoted by $\overline{A}$ and is called the **superior limit** of the sequence $\{x_n\}$; the minimum cluster point is denoted by $\underline{A}$ and is called the **inferior limit** of the sequence $\{x_n\}$. This is denoted as:

$$\overline{\lim_{n\to\infty}}\, x_n = \overline{A}$$

$$\underline{\lim_{n\to\infty}}\, x_n = \underline{A}$$

corollary 2 A bounded sequence necessarily has both a superior and an inferior limit.

Theorem 3 A sequence $\{x_n\}$ converges if and only if its superior and inferior limits exist and are equal. That is,

$$\lim_{n\to\infty} x_n = A \Leftrightarrow \overline{\lim_{n\to\infty}}\, x_n = \underline{\lim_{n\to\infty}}\, x_n = A$$

Therefore, in some limit problems (especially those involving recursive sequences), if we already know that the sequence is bounded, we can take the superior and inferior limits of the recursive formula separately. If the superior and inferior limits are equal, then the original limit exists.

[**e.g.18.1**]Suppose $\{a_n\}$ and $\{b_n\}$ are both bounded sequences, and they satisfy:

$$\lim_{n\to\infty} b_n = b, \quad a_{n+1} + 2a_n = b_n$$

Prove that $\lim_{n\to\infty} a_n$ exists.

Proof: Since $\{a_n\}$ is bounded, let $\overline{A}$ and $\underline{A}$ denote the superior limit and inferior limit of $\{a_n\}$ respectively. Due to the properties of limits and sequences, we have:

$$\overline{\lim_{n\to\infty}}\,(a_n + b_n) = \lim_{n\to\infty} b_n + \overline{\lim_{n\to\infty}}\, a_n$$

Also, from the given recurrence relation:

$$a_{n+2} = b_{n+1} + 2(-a_{n+1})$$

$$-a_{n+1} = -b_n + 2a_n$$

We can derive:

$$
\begin{aligned}
\overline{A} = & \overline{\lim_{n\to\infty}} a_n \\
= & \overline{\lim_{n\to\infty}} a_{n+2} \\
= & b + 2 \overline{\lim_{n\to\infty}} (-a_{n+1}) \\
= & b + 2(-b + 2 \overline{\lim_{n\to\infty}} a_n) \\
= & -b + 4 \overline{\lim_{n\to\infty}} a_n \\
= & -b + 4\overline{A}
\end{aligned}
$$

Therefore:

$$\overline{\lim_{n\to\infty}} a_n = \frac{b}{3}$$

Similarly, we can obtain:

$$\underline{\lim_{n\to\infty}} a_n = \frac{b}{3}$$

Hence:

$$\lim_{n\to\infty} a_n = \frac{b}{3}$$

[e.g.18.2] Let $x_0, y_0 \in \mathbb{R}$. For $n \geq 0$, define:

$$
\begin{cases}
x_{n+1} = \dfrac{1}{x_n^2 + x_n y_n + 2y_n^2 + 1} \\
y_{n+1} = \dfrac{1}{2x_n^2 + x_n y_n + y_n^2 + 1}
\end{cases}
$$

Proof: (1) $\lim_{n\to\infty}(x_n - y_n) = 0$; (2) The sequence $\{x_n\}$ converges.
Proof: (1) Note that $x^2 + xy + y^2 \geq 0$ for all $x, y \in \mathbb{R}$. It is obvious that $x_n, y_n \in [0, 1]$ for all $n \geq 1$. For $n \geq 1$, we have

$$
\begin{aligned}
& |x_{n+1} - y_{n+1}| \\
= & \left| \frac{1}{x_n^2 + x_n y_n + 2y_n^2 + 1} - \frac{1}{2x_n^2 + x_n y_n + y_n^2 + 1} \right| \\
= & \frac{(x_n + y_n)|x_n - y_n|}{(x_n^2 + x_n y_n + 2y_n^2 + 1)(2x_n^2 + x_n y_n + y_n^2 + 1)} \\
\leq & \frac{(x_n + y_n)|x_n - y_n|}{1 + 3x_n^2 + 2x_n y_n + 3y_n^2} \\
\leq & \frac{1}{2}|x_n - y_n|
\end{aligned}
$$

Therefore, $|x_n - y_n| \leq \frac{|x_1 - y_1|}{2^{n-1}}$ for $n \geq 1$. Hence, $\lim_{n\to\infty}(x_n - y_n) = 0$.
(2) Let $L = \overline{\lim}_{n\to\infty} x_n$ and $l = \underline{\lim}_{n\to\infty} x_n$.
Obviously, $0 \leq l \leq L \leq 1$.
Since

$$
\begin{aligned}
& (x_n - y_n)(3x_n + 2y_n) \\
= & 3x_n^2 - x_n y_n - 2y_n^2 \\
= & 4x_n^2 - (x_n^2 + x_n y_n + 2y_n^2)
\end{aligned}
$$

Therefore, $\lim_{n\to\infty}[4x_n^2 - (x_n^2 + x_n y_n + 2y_n^2)] = 0$.
Then,

$$\overline{\lim_{n\to\infty}}(x_n^2 + x_n y_n + 2y_n^2) = 4L^2$$
$$\underline{\lim_{n\to\infty}}(x_n^2 + x_n y_n + 2y_n^2) = 4l^2$$

And since $x_n^2 + x_n y_n + 2y_n^2 = \frac{1}{x_{n+1}} - 1$, we have $4L^2 = \frac{1}{l} - 1$ and $4l^2 = \frac{1}{L} - 1$.

Therefore, $4Ll^2 + L = 4L^2 l + l = 1$. If $L \neq l$, then $4Ll = 1$ and $L + l = 1$, leading to a contradiction since $L = l = \frac{1}{2}$.

Therefore, $L = l$, and hence the sequence $\{x_n\}$ converges.

[e.g.18.3] Let $\beta_n > 0$, and let $f(x)$ be a bounded function on $[-1, 2]$ and Riemann integrable on $[0, 1]$. Prove that:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n} + \beta_n\right) = \int_0^1 f(x)\,dx$$

Proof: First, we introduce a theorem:

Theorem 4 A Riemann integrable function can be approximated by a continuous function, i.e., $\forall\varepsilon > 0$, there exists a continuous function $g(x)$ such that:

$$\left|\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right| < \varepsilon$$

For a detailed proof, see: https://zhuanlan.zhihu.com/p/130394480

Therefore, $\forall\varepsilon > 0$, there exist $f(x)$ and $g(x)$ such that

$$h(x) \le f(x) \le g(x), \quad \int_0^1 g(x)\,dx - \varepsilon \le \int_0^1 f(x)\,dx \le \int_0^1 h(x)\,dx + \varepsilon$$

If $f(x)$ is a continuous function on $[0, 1]$, then it is uniformly continuous on $[0, 1]$. Thus, $\forall\varepsilon > 0, \exists N > 0$ such that when $n > N$,

$$\left|f\left(\frac{k}{n} + \beta_n\right) - f\left(\frac{k}{n}\right)\right| < \varepsilon$$

and

$$\varepsilon = \frac{1}{n}\sum_{k=1}^{n}\varepsilon$$

If $F(x) \in R[-2, 1]$, then:

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \frac{1}{n}\sum_{k=1}^{n} F\left(\frac{k}{n} + \beta_n\right) \\
&\le \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n} + \beta_n\right) \\
&= \int_0^1 g(x)\,dx \\
&\le \int_0^1 F(x)\,dx + \varepsilon
\end{aligned}$$

$$
\begin{aligned}
&\varliminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} F\left(\frac{k}{n}+\beta_n\right) \\
&\geq \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} h\left(\frac{k}{n}+\beta_n\right) \\
&= \int_0^1 h(x)\,dx \\
&\geq \int_0^1 F(x)\,dx - \varepsilon
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^1 F(x)\,dx - \varepsilon \\
&\leq \varliminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} F\left(\frac{k}{n}+\beta_n\right) \\
&\leq \varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} F\left(\frac{k}{n}+\beta_n\right) \\
&\leq \int_0^1 F(x)\,dx + \varepsilon
\end{aligned}
$$

Thus,

$$
\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} F\left(\frac{k}{n}+\beta_n\right) = \int_0^1 F(x)\,dx
$$

However, this is not the end of the proof because f is a bounded function defined on $[-2,1]$ but Riemann integrable on $[0,1]$. Therefore, the value of $\frac{k}{n}+\beta_n$ may exceed $[0,1]$. We also need to consider the case when $\frac{k}{n}+\beta_n < 0$.

In this case, $k \in [1, -n\beta_n] \cup [n(1-\beta_n), n] = S$. Then,

$$
\begin{aligned}
&\left|\lim_{n\to\infty} \frac{1}{n}\sum_{k\in S} f\left(\frac{k}{n}+\beta_n\right)\right| \\
&\leq \sup_{x\in[1,2]} f(x)\cdot \lim_{n\to\infty} \frac{2|n\beta_n|+1}{n} \to 0 \quad (n\to\infty)
\end{aligned}
$$

Therefore,

$$
\lim_{n\to\infty} \frac{1}{n}\sum_{k,-2\leq x<0} f\left(\frac{k}{n}+\beta_n\right) = 0
$$

Thus, we can define

$$
F(x) = \begin{cases} f(x) & \text{if } 0\leq x\leq 1, \\ 0 & \text{if } -2\leq x<0. \end{cases}
$$

Then,

$$\begin{aligned}
\int_0^1 f(x)\,dx &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\left(\frac{k}{n} + \beta_n\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k,0\le x\le 1} f\left(\frac{k}{n} + \beta_n\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k,0\le x\le 1} f\left(\frac{k}{n} + \beta_n\right) + \lim_{n\to\infty} \frac{1}{n} \sum_{k,-2\le x<0} f\left(\frac{k}{n} + \beta_n\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n} + \beta_n\right)
\end{aligned}$$

Q.E.D. □

19 Piecewise Estimation

In some limit problems involving definite integrals, such as

$$\lim_{n\to\infty} \int_a^b f(n,x)dx$$

we often see the approach in the solution is to split the interval, particularly around certain specific points and their neighborhoods. Below, I will provide some typical examples to illustrate how to find the intervals for splitting.

[e.g. 19.1] Find:

$$\lim_{n\to\infty} \int_0^1 x^n dx$$

We observe the integrand and find that when $0 < x < 1$, $x^n \to 0$ as $n \to \infty$. Therefore, we conjecture that the integral value within the neighborhood $[0, 1-\epsilon)$ is small enough to be negligible (i.e., 0), while the integral value within the interval $[1-\epsilon, 1]$ dominates. Thus, we naturally split the integral into two intervals.

$$\lim_{n\to\infty} \int_0^1 x^n dx = \lim_{n\to\infty} \int_0^{1-\varepsilon} x^n dx + \lim_{n\to\infty} \int_{1-\varepsilon}^1 x^n dx$$

Since x^n is monotonically increasing, for the first term,

$$\int_0^{1-\varepsilon} x^n dx \le \int_0^{1-\varepsilon} (1-\varepsilon)^n dx \le (1-\varepsilon)^n \to 0 \text{ as } n\to\infty$$

For the second term:

$$\int_{1-\varepsilon}^1 x^n dx \le \int_{1-\varepsilon}^1 1^n dx = \varepsilon$$

Thus, it is proven that

$$\lim_{n\to\infty} \int_0^1 x^n dx = 0$$

The above example is the simplest and most straightforward. However, it also reflects the essence of this method, which is to find the "extrema points" or "disruptive points" and then split the interval within a small neighborhood of these points. We hope that the integral containing these points has the same value as the original integral (or is 0), and the integral not containing these points is 0 (or has the same value as the original integral). In general, "disruptive points" can be roughly classified into two types:

Definition 1 (Disruptive Points):
Type I: The function $f(n,x)$ has a limit at $x = x_i$ with respect to n that is **not 0** (it can be infinity), and the limit at other points is 0;
Type II: The function $f(n,x)$ has a limit of 0 at $x = x_i$ with respect to n, and the limit at other points is not 0.

When solving the limit

$$\lim_{n\to\infty} \int_a^b f(n,x)dx$$

we generally proceed as follows:
1. We find several disruptive points $x_i \in [a,b]$
(1) If x_i includes an endpoint of the interval $[a,b]$, we can split it into $[a, a+\varepsilon] \bigcup [a+\varepsilon, b]$ or $[a, b-\varepsilon] \bigcup [b-\varepsilon, b]$;
(2) If it includes both endpoints, we can split it into $[a, a+\varepsilon] \bigcup [a+\varepsilon, b-\varepsilon] \bigcup [b-\varepsilon, b]$;
(3) If it includes a point x_i within the open interval, we split it into $[a, x_i-\varepsilon_i] \bigcup [x_i-\varepsilon_i, x_i+\varepsilon_i] \bigcup [x_i+\varepsilon_i, b]$.
2. We sum the integrals over intervals containing disruptive points and over intervals not containing disruptive points.
Then, generally:
(1) When all disruptive points are of Type I (the following equation does not include endpoints; if an endpoint is a disruptive point, it should be added):

$$\lim_{n\to\infty} \sum_{i=1}^{k} \int_{x_i-\varepsilon_i}^{x_i+\varepsilon_i} f(n,x)dx = \lim_{n\to\infty} \int_a^b f(n,x)dx$$

(2) When all disruptive points are of Type II (the following equation does not include endpoints; if an endpoint is a disruptive point, it should be added):

$$\lim_{n\to\infty} \sum_{i=1}^{k} \int_{x_i-\varepsilon_i}^{x_i+\varepsilon_i} f(n,x)dx = 0$$

[e.g.19.2] Suppose $f(x)$ is Riemann integrable on $[a,b]$. $0 \le f(x) < 1, x \in [a,b), f(b) = 1$, find:

$$\lim_{n\to\infty} \int_a^b f^n(x)dx$$

Solution:

$$\int_a^b f^n(x)dx = \int_a^{b-\varepsilon} f^n(x)dx + \int_{b-\varepsilon}^{b} f^n(x)dx$$

For the first term, by the Mean Value Theorem for Integrals, $\exists\ \xi \in (a, b-\varepsilon)\ s.t.$

$$\int_a^{b-\varepsilon} f^n(x)dx \le f^n(\xi)(b-a) \to 0 \quad (n\to\infty)$$

For the second term,

$$\int_{b-\varepsilon}^{b} f^n(x)dx \le \int_{b-\varepsilon}^{b} 1^n dx = \varepsilon$$

Therefore,

$$\lim_{n\to\infty} \int_a^b f^n(x)dx = 0$$

Let's give some specific examples:

[e.g.19.3] Calculate:

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\sin^n xdx$$

Solution:

$$\int_0^{\frac{\pi}{2}}\sin^n xdx=\int_0^{\frac{\pi}{2}-\varepsilon}\sin^n xdx+\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}}\sin^n xdx$$

For the first term:

$$\int_0^{\frac{\pi}{2}-\varepsilon}\sin^n xdx\leq\frac{\pi}{2}\sin^n(\frac{\pi}{2}-\varepsilon)\to 0\quad(n\to\infty)$$

For the second term:

$$\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}}\sin^n xdx\leq\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}}1^n dx=\varepsilon$$

Therefore,

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\sin^n xdx=0$$

[e.g.19.4] $f(x)\in R[0,1]$ is continuous at $x=1$, prove:

$$\lim_{n\to\infty}n\int_0^1 x^n f(x)dx=f(1)$$

Solution: Since $f(x)$ is Riemann integrable on $[0,1]$, it must be bounded, denote the bound as M, and

$$\int_0^1 x^n f(x)dx=\int_0^{1-\varepsilon}x^n f(x)dx+\int_{1-\varepsilon}^1 x^n f(x)dx$$

For the first term, by the Mean Value Theorem for Integrals, $\exists\ \xi\in(0,1-\varepsilon)\ s.t.$

$$|n\int_0^{1-\varepsilon}x^n f(x)dx|=n|\xi^n\int_0^{1-\varepsilon}f(x)dx|\leq n(1-\varepsilon)^n\cdot\sup|f(x)|$$

For the second term,

$$n\int_{1-\varepsilon}^1 x^n f(x)dx=nf(\xi')\int_{1-\varepsilon}^1 x^n dx=f(\xi')\left(\frac{n}{n+1}-n\frac{(1-\varepsilon)^n}{n+1}\right)$$

where $1-\varepsilon<\xi'<1$.

Our goal is to make the first term tend to 0 and the second term tend to $f(1)$ (since x^n dominates and $x=1$ is the maximum value). We only need $n(1-\varepsilon)^n\to 0\quad(n\to\infty)$.

We take $\varepsilon=\frac{1}{\sqrt{n}}$, then

$$\begin{aligned}&\lim_{n\to\infty}n(1-\varepsilon)^n\\&=\lim_{n\to\infty}n(1-\frac{1}{\sqrt{n}})^n\\&=\lim_{n\to\infty}ne^{n\ln(1-\frac{1}{\sqrt{n}})}\\&=\lim_{n\to\infty}e^{n\ln(1-\frac{1}{\sqrt{n}})+\ln n}\\&=\lim_{n\to\infty}e^{n(-\frac{1}{\sqrt{n}}-\frac{1}{2n}+o(\frac{1}{n}))+\ln n}\\&=\lim_{n\to\infty}e^{-\sqrt{n}-\frac{1}{2}+\ln n+o(1)}\\&=0\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty} n\int_0^1 x^n f(x)dx \\
&= \lim_{n\to\infty} n\int_{1-\varepsilon}^1 x^n f(x)dx \\
&= \lim_{n\to\infty} f(\xi')\left(\frac{n}{n+1} - n\frac{(1-\varepsilon)^n}{n+1}\right) \\
&= \lim_{n\to\infty} f(\xi') \\
&= \lim_{\xi'\to 1} f(\xi') \\
&= f(1)
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} n\int_0^1 x^n f(x)dx = f(1)$$

The method of this proof is not difficult to think of, we can use the interval splitting method, but we need to find the appropriate $\varepsilon(n)$ to solve it.

[e.g.19.5] $f(x) \in C[0,1]$, prove that:

$$\lim_{n\to\infty}\int_0^1 \frac{n}{1+n^2x^2}f(x)dx = \lim_{h\to 0^+}\int_0^1 \frac{h}{x^2+h^2}f(x)dx = \frac{\pi}{2}f(0)$$

Proof: We only need to prove that

$$\lim_{h\to 0^+}\int_0^1 \frac{h}{x^2+h^2}f(x)dx = \frac{\pi}{2}f(0)$$

Then, by Heine's theorem, taking the sequence $x_n = \frac{1}{n} \to 0^+ (n \to +\infty)$, we have

$$\lim_{n\to\infty}\int_0^1 \frac{\frac{1}{n}}{x^2+(\frac{1}{n})^2}f(x)dx = \lim_{n\to\infty}\int_0^1 \frac{n}{1+n^2x^2}f(x)dx = \frac{\pi}{2}f(0)$$

Next, we prove that

$$\lim_{h\to 0^+}\int_0^1 \frac{h}{x^2+h^2}f(x)dx = \frac{\pi}{2}f(0)$$

We observe that the integrand takes its maximum value (tending to infinity) as x approaches 0. Therefore, we partition the interval in the neighborhood of $x = 0$.

Then:

$$\int_0^1 \frac{h}{x^2+h^2}f(x)dx = \int_0^\varepsilon \frac{h}{x^2+h^2}f(x)dx + \int_\varepsilon^1 \frac{h}{x^2+h^2}f(x)dx$$

For the first part, applying the mean value theorem for integrals:

$$\int_0^\varepsilon \frac{h}{x^2+h^2}f(x)dx = f(\xi)\int_0^\varepsilon \frac{h}{x^2+h^2}dx = f(\xi)\arctan(\frac{\varepsilon}{h})$$

For the second part,

$$\begin{aligned}&|\int_{\varepsilon}^{1}\frac{h}{x^2+h^2}f(x)dx|\\&\leq \max|f(x)||\int_{\varepsilon}^{1}\frac{h}{x^2+h^2}dx|\\&=M\cdot|\arctan(\frac{1}{h})-\arctan(\frac{\varepsilon}{h})|\end{aligned}$$

Thus, we need to find an ε such that

$$\lim_{h\to 0^+}\arctan(\frac{\varepsilon}{h})=\frac{\pi}{2}$$

In fact, taking $\varepsilon=\sqrt{h}$ suffices, then we have

$$\lim_{h\to 0^+}\int_0^{\varepsilon}\frac{h}{x^2+h^2}f(x)dx=\lim_{h\to 0^+}f(\xi)\arctan(\frac{\varepsilon}{h})=\frac{\pi}{2}f(0)$$

And also

$$\lim_{h\to 0^+}M\cdot[\arctan(\frac{1}{h})-\arctan(\frac{\varepsilon}{h})]=0$$

Therefore

$$\begin{aligned}&\lim_{h\to 0^+}\int_0^1\frac{h}{x^2+h^2}f(x)dx\\&=\lim_{h\to 0^+}\int_0^{\varepsilon}\frac{h}{x^2+h^2}f(x)dx+\lim_{h\to 0^+}\int_{\varepsilon}^1\frac{h}{x^2+h^2}f(x)dx\\&=\boxed{\frac{\pi}{2}f(0)}\end{aligned}$$

Sometimes, the problem may not necessarily require finding a limit, but rather comparing sizes, etc. Partitioning the interval can also be considered.

[e.g.19.6] Suppose that:

$$a_n=\int_0^{\frac{\pi}{2}}t\left|\frac{\sin(nt)}{\sin(t)}\right|^3dt$$

Prove that $\sum\frac{1}{a_n}$ diverges.

Proof: We observe that $\left|\frac{\sin(nt)}{\sin(t)}\right|^3\to n^3$ as $t\to 0^+$ (since n can take infinitely large values, we consider this point as the "disruptive point" or "point of maximum value"). (The function is bounded at other points), therefore we consider:

$$\int_0^{\frac{\pi}{2}}t\left|\frac{\sin(nt)}{\sin(t)}\right|^3dt=\int_0^{\varepsilon}t\left|\frac{\sin(nt)}{\sin(t)}\right|^3dt+\int_{\varepsilon}^{\frac{\pi}{2}}t\left|\frac{\sin(nt)}{\sin(t)}\right|^3dt$$

Since we are not seeking a limit, both integrals can be bounded but will not be zero.

To prove that $\sum\frac{1}{a_n}$ diverges, we only need to prove that $a_n<kn$ (where $k>0$), then $\sum\frac{1}{a_n}>\sum\frac{1}{kn}$ which diverges.

For the first part, we first prove the fact that $|\sin nt|\leq n|\sin t|$.

We use mathematical induction to prove this:

1. When $n=1$, it is obviously true;
2. Assume it holds for $n\leq k$;

Then for $n = k + 1$:

$$\begin{aligned}
|\sin(k+1)x| &= |\sin(kx+x)| \\
&= |\sin kx \cos x + \sin x \cos kx| \\
&\le |\sin kx \cos x| + |\sin x \cos kx| \\
&\le |\sin kx||\cos x| + |\sin x||\cos kx| \\
&\le |\sin kx| + |\sin x| \\
&\le k|\sin x| + |\sin x| \\
&= (k+1)|\sin x|
\end{aligned}$$

Therefore,

$$\int_0^\varepsilon t \left|\frac{\sin(nt)}{\sin(t)}\right|^3 dt \le n^3 \int_0^\varepsilon t dx = \frac{\varepsilon^2 n^3}{2}$$

For the second part, since $\sin x \ge \frac{2}{\pi}x$ for $0 \le x \le \frac{\pi}{2}$, then

$$\begin{aligned}
&\int_\varepsilon^{\frac{\pi}{2}} t \left|\frac{\sin(nt)}{\sin(t)}\right|^3 dt \\
&\le \int_\varepsilon^{\frac{\pi}{2}} t \left|\frac{1}{\sin(t)}\right|^3 dt \\
&\le \int_\varepsilon^{\frac{\pi}{2}} t \left(\frac{\pi}{2t}\right)^3 dt \\
&= \frac{\pi^3}{8}\left(\frac{1}{\varepsilon} - \frac{2}{\pi}\right) \\
&< \frac{\pi^3}{8\varepsilon}
\end{aligned}$$

Therefore, we need to find an ε such that

$$\frac{\varepsilon^2 n^3}{2} + \frac{\pi^3}{8\varepsilon} < kn$$

Here we choose $\varepsilon = \frac{1}{n}$, then

$$\frac{\varepsilon^2 n^3}{2} + \frac{\pi^3}{8\varepsilon} < 10n$$

Thus,

$$\sum \frac{1}{a_n} > \frac{1}{10} \sum \frac{1}{n}$$

Therefore, $\sum \frac{1}{a_n}$ diverges.

[e.g.19.7] Find:

$$\lim_{n\to\infty}\int_0^1 2x\cdot\arctan(nx)\cdot e^{x^2}dx$$

Solution: The integrand tends to 0 only when $x\to 0$, so the disruptive point is $x=0$. Then

$$\begin{aligned}&\int_0^1 2x\cdot\arctan(nx)\cdot e^{x^2}dx\\&=\int_0^{\frac{1}{\sqrt{n}}} 2x\cdot\arctan(nx)\cdot e^{x^2}dx+\int_{\frac{1}{\sqrt{n}}}^1 2x\cdot\arctan(nx)\cdot e^{x^2}dx\end{aligned}$$

So

$$\begin{aligned}0\le I_1&=\int_0^{\frac{1}{\sqrt{n}}} 2x\cdot\arctan(nx)\cdot e^{x^2}dx\\&\le\int_0^{\frac{1}{\sqrt{n}}} 2x\cdot\arctan(n\frac{1}{\sqrt{n}})\cdot e^{x^2}dx\\&=\arctan(\sqrt{n})(e^{\frac{1}{n}}-1)\to 0\quad(n\to\infty)\end{aligned}$$

$$\begin{aligned}I_2&=\int_{\frac{1}{\sqrt{n}}}^1 2x\cdot\arctan(nx)\cdot e^{x^2}dx\\&=\arctan(n\xi)\int_{\frac{1}{\sqrt{n}}}^1 2x\cdot e^{x^2}dx\\&=\arctan(n\xi)(e-e^{\frac{1}{n}})\to\frac{\pi}{2}(e-1)\quad(n\to\infty)\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\int_0^1 2x\cdot\arctan(nx)\cdot e^{x^2}dx=\frac{\pi}{2}(e-1)$$

20 Arzela's Dominated Convergence Theorem

In the previous section, we used the method of dividing the interval and estimating piecewise to calculate limits of the form $\lim_{n\to\infty}\int_a^b f_n(x)dx$. Naturally, we would think that if the limit and integral can be exchanged in order, it would be more convenient to first find the inner limit, so that

$$\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^b f(x)dx.$$

Below, we introduce several definitions and theorems to ensure that we can operate in this way:

Definition 1 (Uniformly Bounded) Let $\{f_n(x)\}$ be a sequence of functions. If there exists $M>0$ such that for all n and $x\in D$,

$$|f_n(x)|\le M,$$

then $\{f_n(x)\}$ is said to be uniformly bounded on D.

Definition 2 (Pointwise Convergence) Let $\{f_n(x)\}$ be a sequence of functions. If for any $x_0 \in D$, $\lim_{n\to\infty} f_n(x_0) = f(x_0)$, i.e., $\forall x_0 \in D$ (given x_0 beforehand), $\forall \varepsilon > 0$, there exists $N > 0$ such that when $n > N$,

$$|f_n(x_0) - f(x_0)| < \varepsilon,$$

then the function sequence $f_n(x)$ converges pointwise to $f(x)$.

Definition 3 (Uniform Convergence) Let $\{f_n(x)\}$ be a sequence of functions. If $\forall \varepsilon > 0$, there exists $N > 0$ such that when $n > N$, for $\forall x \in D$,

$$|f_n(x) - f(x)| < \varepsilon,$$

then the function sequence $f_n(x)$ converges uniformly to $f(x)$.

Theorem 1 Let $\{f_n(x)\}$ converge uniformly to $f(x)$ on the closed interval $I = [a, b]$, and let each term be continuous. Then:
1. $f(x)$ is continuous on I;
2.
$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx;$$
3.
$$\frac{d}{dx}\left[\lim_{n\to\infty} f_n(x)\right] = \lim_{n\to\infty} \frac{d}{dx} f_n(x).$$

Theorem 2 (Arzela's Dominated Convergence Theorem) Let $\{f_n(x)\}$ (with $x \in D = [a, b]$) be a sequence of functions satisfying:
1. For any n, x, $f_n(x)$ is Riemann integrable;
2. $\{f_n(x)\}$ is uniformly bounded on D;
3. $\lim_{n\to\infty} f_n(x) = f(x)$ (uniform convergence is not required), and $f(x)$ is Riemann integrable on D.

Then
$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

[**e.g.20.1**] Solve for:

$$\lim_{n\to\infty} \int_0^1 x^n dx$$

Solution: Since

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & , \text{ if } 0 \le x < 1 \\ 1 & , \text{ if } x = 1 \end{cases}$$

therefore

$$\lim_{n\to\infty} \int_0^1 x^n dx = \lim_{n\to\infty} \int_0^{1-\varepsilon} x^n dx + \lim_{n\to\infty} \int_{1-\varepsilon}^1 x^n dx = I_1 + I_2$$

For the first term

$$I_1 = \lim_{n\to\infty} \int_0^{1-\varepsilon} x^n dx = \int_0^{1-\varepsilon} \lim_{n\to\infty} x^n dx = 0$$

For the second term

$$I_2 = \lim_{n\to\infty} \int_{1-\varepsilon}^1 x^n dx \le \lim_{n\to\infty} \int_{1-\varepsilon}^1 1^n dx = \varepsilon$$

Thus

$$\lim_{n\to\infty}\int_0^1 x^n dx = 0$$

Through this problem, we can see that if $\lim_{n\to\infty} f_n(x) = f(x)$, then the discontinuity of $f(x)$ at the endpoints does not affect the integral value (provided it is integrable). Therefore, for such problems in the future, we do not need to consider the endpoints or split the interval.

[**e.g.20.2**] Solve for:

$$\lim_{n\to\infty}\int_0^1 \frac{dt}{(1+t^4)^n}$$

Solution:

$$\lim_{n\to\infty}\int_0^1 \frac{dt}{(1+t^4)^n} = \int_0^1 \lim_{n\to\infty}\frac{dt}{(1+t^4)^n} = \int_0^1 0dt = \boxed{0}$$

[**e.g.20.3**] Solve for:

$$\lim_{n\to\infty} n\int_1^a \frac{dx}{1+x^n} \quad (a>1)$$

Solution: Let $t=\frac{1}{x}, dx=-\frac{1}{t^2}dt$, then

$$\lim_{n\to\infty} n\int_1^a \frac{dx}{1+x^n} = \lim_{n\to\infty} n\int_1^{\frac{1}{a}} \frac{1}{1+\frac{1}{t^n}}\left(-\frac{1}{t^2}dt\right) = \lim_{n\to\infty}\int_{\frac{1}{a}}^1 \frac{nt^{n-2}}{1+t^n}dt = \lim_{n\to\infty}\int_{\frac{1}{a}}^1 \frac{nt^{n-1}}{t(1+t^n)}dt$$

Because

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\frac{1}{a}}^1 \frac{nt^{n-1}}{t(1+t^n)}dt \\
&= \lim_{n\to\infty}\int_{\frac{1}{a}}^1 \frac{1}{t}d[\ln(1+t^n)] \\
&= \ln 2 - \lim_{n\to\infty} a\ln\left(1+\frac{1}{a^n}\right) + \lim_{n\to\infty}\int_{\frac{1}{a}}^1 \frac{\ln(1+t^n)}{t^2}dt \\
&= \ln 2 + \lim_{n\to\infty}\int_{\frac{1}{a}}^1 \frac{\ln(1+t^n)}{t^2}dt
\end{aligned}$$

When $\frac{1}{a}\le t<1$,

$$\lim_{n\to\infty}\frac{\ln(1+t^n)}{t^2} = \lim_{n\to\infty}\frac{t^n}{t^2} = 0$$

Therefore

$$\lim_{n\to\infty}\int_{\frac{1}{a}}^1 \frac{\ln(1+t^n)}{t^2}dt = \int_{\frac{1}{a}}^1 \lim_{n\to\infty}\frac{\ln(1+t^n)}{t^2}dt = 0$$

Thus

$$\lim_{n\to\infty} n\int_1^a \frac{dx}{1+x^n} = \boxed{\ln 2}$$

[**e.g.20.4**] Solve for:

$$\lim_{n\to\infty}\int_0^1 2x\cdot\arctan(nx)\cdot e^{x^2}dx$$

Solution:

$$
\begin{aligned}
&\lim_{n\to\infty}\int_0^1 2x\cdot\arctan(nx)\cdot e^{x^2}dx\\
&=\int_0^1 \lim_{n\to\infty} 2x\cdot\arctan(nx)\cdot e^{x^2}dx\\
&=\int_0^1 2x\cdot\frac{\pi}{2}\cdot e^{x^2}dx\\
&=\frac{\pi}{2}\int_0^1 e^{x^2}d(x^2)\\
&=\boxed{\frac{\pi}{2}(e-1)}
\end{aligned}
$$

21 Laplace Method

When solving limits of the form $\lim_{n\to\infty}\int_a^b f_n(x)dx$, one approach is to find the ”maximum points” or ”points of discontinuity”, and then manually split the interval to isolate these points, calculating the limit for each segment separately.

Another method is to use the Arzela-Ascoli Theorem to exchange the order of integration and limit. Among these, for a special type of limit like

$$\lim_{n\to\infty}\int_a^b \varphi(x)e^{nh(x)}dx \quad (*)$$

we can use the Laplace Method to solve it.

The Laplace Method is an effective method for limits of integrals, essentially identifying the ”concentration point of order,” i.e., which part dominates as n approaches infinity. This theorem primarily addresses limits of the above type:

If we have an integral limit like

$$\lim_{n\to\infty}\int_a^b \varphi(x)f^n(x)dx \quad (f(x)>0)$$

we can let $h(x)=\ln f(x)$ to convert it into a limit of type $(*)$. The following outlines the main content of the Laplace Method:

Theorem 1 (Laplace Method): Suppose $\varphi(x)$ and $h(x)$ are defined on $[a,b]$, and:
(1) There exists $n_0\in\mathbb{N}$, such that for all $n\geq n_0$, $\varphi(x)e^{nh(x)}$ is integrable on $[a,b]$;
(2) 1. There exists a unique point $\xi\in(a,b)$ such that $h(\xi)$ is the maximum value of $h(x)$ on $[a,b]$, 2. For any closed interval $[\alpha,\beta]$ that does not contain ξ, i.e., $\xi\notin[\alpha,\beta]\subset[a,b]$, we have $\sup_{x\in[\alpha,\beta]}h(x)<h(\xi)$;
(3) Within a small neighborhood of ξ, $h''(x)$ is continuous and $h''(\xi)<0$;
(4) $\varphi(x)$ is continuous at $x=\xi$ and $\varphi(\xi)\neq 0$.
Then, as $n\to\infty$, we have

$$\int_a^b \varphi(x)e^{nh(x)}dx \sim \varphi(\xi)\sqrt{-\frac{2\pi}{nh''(\xi)}}e^{nh(\xi)}$$

Let’s explain these conditions first.

Firstly, it is obvious that $\varphi(x)$ and $h(x)$ are defined on $[a,b]$ because the range of x in the definite integral is $[a,b]$;

Secondly, condition (1) requires that $\varphi(x)e^{nh(x)}$ be integrable for sufficiently large n, as we are seeking the limit and can thus ignore the finite number of terms at the beginning;

Condition (2) embodies the idea of piecewise estimation but requires that $h(x)$ has only one maximum point that is not at an endpoint. The second part addresses certain discontinuous functions that have only one non-endpoint maximum but may have points of discontinuity of the first kind. For example,

$$h(x) = \begin{cases} x, & 0 \le x \le 1 \\ -x+2, & 1 < x \le 2 \\ x-2, & 2 < x < 3 \\ 0, & x = 3 \\ -x+4, & 3 < x \le 4 \end{cases}$$

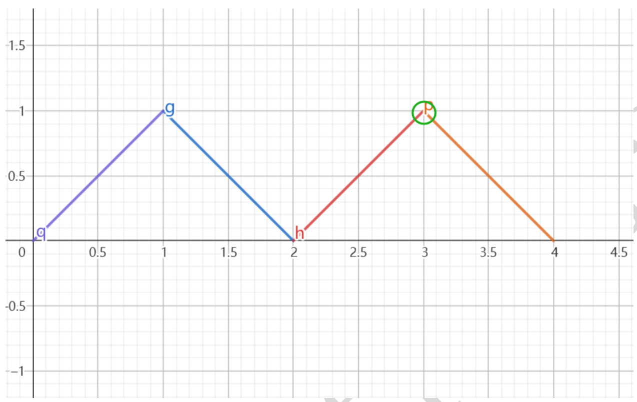

Figure 1: Graph of $h(x)$

$h(x)$ has only one maximum point $x = 1$ on $[0, 4]$, but on $[2, 4]$, $\sup_{x\in[2,4]} h(x) = 1$, which does not satisfy the second part of condition (2). Conditions (3) and (4) are required to make the intermediate derivations and results meaningful.

For convenience, let's recall the O notation. It is defined as:

If $g(x) > 0$, there exists $A > 0$ such that $|f(x)| \le Ag(x) \quad (x \in D)$, then we write

$$f(x) = O(g(x))$$

.

Property: If $|f(x)| \le O(g(x))$, then $f(x) = O(g(x))$.

Proof: Since

$$|f(x)| \le O(g(x)) \le Ag(x)$$

it follows that

$$f(x) = O(g(x))$$

Below is the proof of the theorem:

Proof:

Firstly, we determine an n_0. In fact, we only need to take $n_0 = 0$, because if it has already been proven for $n_0 = 0$, then the theorem holds for all n such that

$$n \ge n_1 \ge n_0 = 0$$

.

We first partition the integral interval. Since ξ is the point of maximum, we isolate the point ξ.

Since this differs somewhat from our previous discussion (which was about finding limits, while this is about proving equivalence, which is equivalent to finding the order), we need to partition the interval more precisely, i.e., near the ξ point. Therefore, it will be more convenient for us to discuss $[\xi - \varepsilon, \xi + \varepsilon]$.

And we need to use the derivative of $h(x)$ in the proof, so we need to partition an additional interval to ensure that $h(x)$ is differentiable (or second-order differentiable) in that interval.

Therefore, we choose a small $\delta > 0$ such that $h''(x) \leq -k < 0$ for $x \in [\xi - \delta, \xi + \delta]$ (second-order differentiable with an upper bound).

We calculate

$$\int_a^b \varphi(x)e^{n[h(x)-h(\xi)]}dx = \left(\int_a^{\xi-\delta} + \int_{\xi-\delta}^{\xi-n^{-\frac{2}{5}}} + \int_{\xi-n^{-\frac{2}{5}}}^{\xi+n^{-\frac{2}{5}}} + \int_{\xi+n^{-\frac{2}{5}}}^{\xi+\delta} + \int_{\xi+\delta}^{b}\right) \varphi(x)e^{n[h(x)-h(\xi)]}dx$$

Denote each integral as I_1, I_2, I_3, I_4, I_5. Then $I = I_1 + I_2 + I_3 + I_4 + I_5$ (we will see why we partition the interval this way later).

Let

$$m' = \sup_{x\in[a,\xi-\delta]} [h(x) - h(\xi)] < 0$$

then

$$I_1 = \int_a^{\xi-\delta} \varphi(x)e^{n[h(x)-h(\xi)]}dx$$

Then $e^{n[h(x)-h(\xi)]} \leq 1 \cdot e^{nm'}$, i.e., $e^{n[h(x)-h(\xi)]} = O(e^{nm'})$. And $\int_a^{\xi-\delta} \varphi(x)dx$ is bounded, i.e., $\int_a^{\xi-\delta} \varphi(x) \leq M$. Then

$$I_1 = O(e^{nm'}) \int_a^{\xi-\delta} \varphi(x)dx = O(e^{nm'}) \cdot O(1) = O(e^{nm'})$$

Similarly, if we let $m'' = \sup_{x\in[\xi+\delta,b]}[h(x) - h(\xi)] < 0$, then $I_5 = O(e^{nm''})$.

Next, we discuss I_2. On $[\xi - \delta, \xi - n^{-\frac{2}{5}}]$, $h(x)$ is second-order differentiable. Since $h(\xi)$ is the maximum and continuous near ξ, $h'(\xi) = 0$.

$$h(x) = h(\xi) + \frac{h''(\eta)}{2}(x - \xi)^2, \quad \eta = \xi + \theta(x - \xi), \quad 0 < \theta < 1$$

Combined with the previous condition $h''(x) \leq -k < 0$ for $x \in [\xi - \delta, \xi + \delta]$, we have

$$h(x) - h(\xi) = \frac{h''(\eta)}{2}(x - \xi)^2 \leq -\frac{k}{2}(x - \xi)^2$$

$$\begin{aligned}
I_2 &= \int_{\xi-\delta}^{\xi-n^{-\frac{2}{5}}} \varphi(x)e^{n[h(x)-h(\xi)]}dx \\
&\leq \int_{\xi-\delta}^{\xi-n^{-\frac{2}{5}}} \varphi(x)e^{-n\frac{k}{2}(x-\xi)^2}dx \\
&\leq \int_{\xi-\delta}^{\xi-n^{-\frac{2}{5}}} \varphi(x)e^{-n\frac{k}{2}n^{-\frac{4}{5}}}dx \\
&= \int_{\xi-\delta}^{\xi-n^{-\frac{2}{5}}} \varphi(x)e^{-\frac{k}{2}n^{\frac{1}{5}}}dx \\
&= O\left(e^{-\frac{k}{2}n^{\frac{1}{5}}}\right)
\end{aligned}$$

Similarly, $I_4 = O(e^{-\frac{k}{2}n^{\frac{1}{5}}})$.

Next, we estimate I_3. By the mean value theorem for integrals, we have

$$I_3 = \int_{\xi-n^{-\frac{2}{5}}}^{\xi+n^{-\frac{2}{5}}} \varphi(x)e^{n[h(x)-h(\xi)]}dx = \varphi(\xi_n)\int_{\xi-n^{-\frac{2}{5}}}^{\xi+n^{-\frac{2}{5}}} e^{n[h(x)-h(\xi)]}dx$$

Since $\lim_{n\to\infty}\varphi(\xi_n)=\varphi(\xi)$, we have

$$\varphi(\xi_n)=\varphi(\xi)+\varphi(\xi)\cdot o(1)=\varphi(\xi)(1+o(1))$$

Thus,

$$I_3=\varphi(\xi)(1+o(1))\int_{\xi-n^{-\frac{2}{5}}}^{\xi+n^{-\frac{2}{5}}} e^{n[h(x)-h(\xi)]}dx$$

Let $S=[\xi-n^{-\frac{2}{5}},\xi+n^{-\frac{2}{5}}]$, and

$$a=\min_{x\in S}h''(x),\quad A=\max_{x\in S}h''(x)$$

Then

$$a=h''(\xi)(1+o(1)),\quad A=h''(\xi)(1+o(1))$$

Since

$$\frac{a}{2}(x-\xi)^2\leq h(x)-h(\xi)=\frac{h''(\eta)}{2}(x-\xi)^2\leq\frac{A}{2}(x-\xi)^2$$

we have

$$\int_{\xi-n^{-\frac{2}{5}}}^{\xi+n^{-\frac{2}{5}}} e^{n\frac{a}{2}(x-\xi)^2}dx\leq\int_{\xi-n^{-\frac{2}{5}}}^{\xi+n^{-\frac{2}{5}}} e^{n[h(x)-h(\xi)]}dx\leq\int_{\xi-n^{-\frac{2}{5}}}^{\xi+n^{-\frac{2}{5}}} e^{n\frac{A}{2}(x-\xi)^2}dx$$

Calculating the left-hand integral, let $-\frac{1}{2}na(x-\xi)^2=y^2$, so $dx=\sqrt{-\frac{2}{na}}dy$. Therefore,

$$\begin{aligned}I_{left}&=\int_{\xi-n^{-\frac{2}{5}}}^{\xi+n^{-\frac{2}{5}}} e^{n\frac{a}{2}(x-\xi)^2}dx\\&=\int_{-\sqrt{-\frac{a}{2}}\cdot n^{\frac{1}{10}}}^{\sqrt{-\frac{a}{2}}\cdot n^{\frac{1}{10}}} e^{-y^2}\sqrt{-\frac{2}{na}}dy\\&=\sqrt{-\frac{2}{na}}\left(\int_{-\infty}^{+\infty}-\int_{\sqrt{-\frac{a}{2}}\cdot n^{\frac{1}{10}}}^{+\infty}-\int_{-\infty}^{-\sqrt{-\frac{a}{2}}\cdot n^{\frac{1}{10}}}\right)e^{-y^2}dy\end{aligned}$$

Since

$$\int_{-\infty}^{+\infty} e^{-y^2}dy=\sqrt{\pi}$$

when n is sufficiently large, the last two terms tend to 0. Therefore,

$$\left(-\int_{\sqrt{-\frac{a}{2}}\cdot n^{\frac{1}{10}}}^{+\infty}-\int_{-\infty}^{-\sqrt{-\frac{a}{2}}\cdot n^{\frac{1}{10}}}\right)e^{-y^2}dy=o(1)=o(1)\cdot\sqrt{\pi}$$

Then

$$I_{left}=\sqrt{-\frac{2}{na}}(1+o(1))\sqrt{\pi}=\sqrt{-\frac{2\pi}{nh''(\xi)}}(1+o(1)),\quad(n\to\infty)$$

Similarly,

$$I_{right}=\sqrt{-\frac{2\pi}{nh''(\xi)}}(1+o(1)),\quad n\to\infty$$

Thus,

$$I_3=\varphi(\xi)\sqrt{-\frac{2\pi}{nh''(\xi)}}(1+o(1)),\quad n\to\infty$$

Then

$$I = \int_a^b \varphi(x)e^{n[h(x)-h(\xi)]}dx = \varphi(\xi)\sqrt{-\frac{2\pi}{nh''(\xi)}}(1+o(1)) + O(e^{nm'}) + O(e^{nm''}) + O(e^{-\frac{k}{2}n^{\frac{1}{5}}})$$

Since $m', m'' < 0$, we have

$$O(e^{nm'}) + O(e^{nm''}) + O(e^{-\frac{k}{2}n^{\frac{1}{5}}}) \to 0 \quad (n \to \infty)$$

Thus,

$$\int_a^b \varphi(x)e^{n[h(x)-h(\xi)]}dx = \varphi(\xi)\sqrt{-\frac{2\pi}{nh''(\xi)}}(1+o(1)) \quad (n \to \infty)$$

Multiplying both sides by $e^{h(\xi)}$ gives

$$\int_a^b \varphi(x)e^{nh(x)}dx \sim \varphi(\xi)\sqrt{-\frac{2\pi}{nh''(\xi)}}e^{nh(\xi)} \quad (n \to \infty)$$

This completes the proof of the theorem.

Note: If a or b is ∞, it is only necessary that $\varphi(x)e^{nh(x)}$ (for $n \geq n_0$) be integrable over the infinite interval.

We use this theorem to derive **Stirling's formula**.

[**e.g.21.1**]Proof:

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n \quad (n \to \infty)$$

Proof: Consider

$$\Gamma(n+1) = \int_0^\infty y^n e^{-y}dy = n^{n+1}\int_0^\infty (xe^{-x})^n dx \quad \text{(substitution } y = nx)$$

Then we can take

$$a = 0, \quad b = +\infty, \quad \varphi(x) = 1, \quad h(x) = \ln x - x, \quad \xi = 1$$

Therefore,

$$n! \sim n^{n+1}\sqrt{\frac{2\pi}{n}}e^{-n} = \sqrt{2n\pi}\left(\frac{n}{e}\right)^n \quad (n \to \infty)$$

But in practical problems, we often encounter situations where the maximum value occurs at an endpoint. We need to slightly modify the conditions and conclusions. Below are two theorems that address the situation where the maximum value occurs at an endpoint. We only consider the case where the maximum value occurs at the left endpoint $x = a$, and the case where the maximum value occurs at the right endpoint can be converted to the left endpoint through variable substitution.

Theorem 2

Suppose $\varphi(x)$ and $h(x)$ are defined on $[a, b]$, and:
(1) $\exists n_0 \in \mathbb{N}$, such that for $\forall n \geq n_0$, $\varphi(x)e^{nh(x)}$ is integrable on $[a, b]$;
(2) 1. $h(a)$ is the maximum value of $h(x)$ on $[a, b]$,
2. For any closed interval $[\alpha, \beta]$ that does not contain a, we have $\sup_{x\in[\alpha,\beta]} h(x) < h(a)$;
(3) In some right neighborhood of a, $h''(x)$ and $\varphi(x)$ are both continuous;
(4) $h'(a) = 0, h''(a) < 0$, and $\varphi(a) \neq 0$.
Then as $n \to \infty$, we have

$$\int_a^b \varphi(x)e^{nh(x)}dx \sim \varphi(a)\sqrt{-\frac{\pi}{2nh''(a)}}e^{nh(a)}$$

Sometimes, $h'(a)=0$ may not be satisfied, so we have the following theorem:

Theorem 3
Suppose $\varphi(x)$ and $h(x)$ are defined on $[a,b]$, and:
(1) $\exists n_0 \in \mathbb{N}$, such that for $\forall n \geq n_0$, $\varphi(x)e^{nh(x)}$ is integrable on $[a,b]$;
(2) 1. $h(a)$ is the maximum value of $h(x)$ on $[a,b]$, 2. For any closed interval $[\alpha,\beta]$ that does not contain a, we have $\sup_{x\in[\alpha,\beta]} h(x) < h(a)$;
(3) In some right neighborhood of a, $h'(x)$ and $\varphi(x)$ are both continuous;
(4) $h'(a) < 0$ and $\varphi(a) \neq 0$.
Then as $n \to \infty$, we have

$$\int_a^b \varphi(x)e^{nh(x)}dx \sim -\frac{\varphi(a)}{nh'(a)}e^{nh(a)}$$

[e.g.21.2] Calculate:

$$\lim_{n\to\infty}\int_0^1 x^n dx$$

Solution: Let $t = 1-x$, then $dx = -dt$, thus

$$\int_0^1 x^n dx = \int_0^1 (1-t)^n dt = \int_0^1 (1-x)^n dx$$

Let

$$\varphi(x) = 1, h(x) = \ln(1-x), \xi = a = 0, h(0) = 0, h'(0) = -1 < 0$$

Then using Theorem 3, we immediately obtain

$$\int_0^1 (1-x)^n dx \sim \frac{1}{n} \quad (n \to \infty)$$

Therefore

$$\lim_{n\to\infty}\int_0^1 x^n dx = \boxed{0}$$

[e.g.21.3] Calculate:

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}} \sin^n x dx$$

Solution: Let $t = \frac{\pi}{2} - x$, then $dx = -dt$, thus

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n t dt = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

Let

$$\varphi(x) = 1, h(x) = \ln(\cos x), \xi = a = 0, h(0) = 0, h'(0) = 0, h''(0) = -1 < 0$$

Then using Theorem 2, we immediately obtain

$$\int_0^{\frac{\pi}{2}} \cos^n x dx \sim \sqrt{\frac{\pi}{2n}} \quad (n \to \infty)$$

Therefore

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}} \sin^n x dx = \boxed{0}$$

[e.g.21.4] Calculate:

$$\lim_{n\to\infty} \sqrt{n}\int_0^{\infty} \frac{\cos x}{(1+x^2)^n}dx$$

Solution: Let

$$a=0, b=\infty, \varphi(x)=\cos x, h(x)=-\ln(1+x^2), \xi=a=0, h'(a)=0, h''(a)=-2$$

Then using Theorem 2, we immediately obtain

$$\int_0^{\infty} \frac{\cos x}{(1+x^2)^n}dx \sim \frac{1}{2}\sqrt{\frac{\pi}{n}} \quad (n\to\infty)$$

Therefore

$$\lim_{n\to\infty} \sqrt{n}\int_0^{\infty} \frac{\cos x}{(1+x^2)^n}dx = \lim_{n\to\infty} \sqrt{n}\frac{1}{2}\sqrt{\frac{\pi}{n}} = \boxed{\frac{\sqrt{\pi}}{2}}$$

[e.g.21.5] Calculate:

$$\lim_{n\to\infty} \sqrt{n}\int_0^{1} \left(\frac{\sin x}{x}\right)^n dx$$

Solution: Let

$$a=0, b=1, \varphi(x)=1, h(x)=\ln(\frac{\sin x}{x}), \xi=a=0, h'(a+0)=0, h''(a+0)=-\frac{1}{3}$$

Since it is undefined at $x=0$, here we use the right limit at this point, and Theorem 2 can still be applied, yielding:

$$\int_0^{1} (\frac{\sin x}{x})^n dx \sim \frac{1}{2}\sqrt{\frac{6\pi}{n}} \quad (n\to\infty)$$

Therefore

$$\lim_{n\to\infty} \sqrt{n}\int_0^{1} (\frac{\sin x}{x})^n dx = \lim_{n\to\infty} \sqrt{n}\frac{1}{2}\sqrt{\frac{6\pi}{n}} = \boxed{\frac{\sqrt{6\pi}}{2}}$$

[e.g.21.6] Calculate:

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n}\frac{n^k}{k!}}{e^n} = \lim_{n\to\infty} \frac{1+\frac{n}{1!}+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}}{e^n}$$

Solution: Since the Laplace method can only solve limits involving integrals, this problem only has a limit but no integral, so we need to construct one. Observing the numerator, it is actually the Taylor expansion of e^n up to the n-th term. Therefore, we naturally think of using the Taylor formula with an integral remainder term. Namely:

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt$$

First take $f(x)=e^x$, then

$$e^x = 1+x+\frac{1}{2}x^2+\cdots+\frac{x^k}{n!}+\frac{1}{n!}\int_0^x e^t(x-t)^n dt$$

Substituting $x=n$, we get

$$e^n = 1+\frac{n}{1!}+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}+\frac{1}{n!}\int_0^n e^t(n-t)^n dt$$

Thus

$$\begin{aligned}&\lim_{n\to\infty}\frac{1+\frac{n}{1!}+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}}{e^n}\\&=\lim_{n\to\infty}\frac{e^n-\frac{1}{n!}\int_0^n e^t(n-t)^n dt}{e^n}\\&=1-\lim_{n\to\infty}\frac{1}{n!}\frac{\int_0^n e^t(n-t)^n dt}{e^n}\end{aligned}$$

We consider

$$\int_0^n e^t(n-t)^n dt$$

Let $t=nx, dt=ndx$, then

$$\int_0^n e^t(n-t)^n dt=n^{n+1}\int_0^1 e^{nx}(1-x)^n dx=n^{n+1}\int_0^1 e^{n[x+\ln(1-x)]}dx$$

Take

$$a=0, b=1, \varphi(x)=1, h(x)=x+\ln(1-x), \xi=a=0, h'(a)=0, h''(a)=-1$$

Then by Theorem 2, we get

$$\int_0^1 e^{n[x+\ln(1-x)]}dx\sim\sqrt{\frac{\pi}{2n}}(n\to\infty)$$

So

$$\lim_{n\to\infty}\frac{1}{n!}\frac{\int_0^n e^t(n-t)^n dt}{e^n}=\lim_{n\to\infty}\frac{n^{n+1}}{n!e^n}\sqrt{\frac{\pi}{2n}}$$

By Stirling's formula $n!\sim\sqrt{2n\pi}(\frac{n}{e})^n$ $(n\to\infty)$, we get

$$\lim_{n\to\infty}\frac{n^{n+1}}{n!e^n}\sqrt{\frac{\pi}{2n}}=\lim_{n\to\infty}\frac{n^{n+1}}{\sqrt{2n\pi}(\frac{n}{e})^n e^n}\sqrt{\frac{\pi}{2n}}=\frac{1}{2}$$

Therefore

$$\lim_{n\to\infty}\frac{1+\frac{n}{1!}+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}}{e^n}=1-\frac{1}{2}=\boxed{\frac{1}{2}}$$

[e.g.21.7] Let $f(x)=1-x^2+x^3$ and calculate the following limit:

$$\lim_{n\to\infty}\frac{\int_0^1 f^n(x)\ln(x+2)dx}{\int_0^1 f^n(x)dx}$$

Solution: We find that "$h(x)$" has two maximum points $x=0, x=1$ within the integration interval, so we need to split the interval. We can split the interval $[0,1]=[0,\frac{1}{2}]\cup[\frac{1}{2},1]$, thus:

$$\int_0^1 f^n(x)\ln(x+2)dx=\int_0^{\frac{1}{2}} f^n(x)\ln(x+2)dx+\int_{\frac{1}{2}}^1 f^n(x)\ln(x+2)dx=I_1+I_2$$

For I_1: take

$$a=0, b=\frac{1}{2}, \varphi(x)=\ln(x+2), h(x)=\ln(1-x^2+x^3), \xi=a=0, h'(0)=0, h''(0)=$$

Therefore, by Theorem 2, we get

$$I_1\sim\ln 2\sqrt{\frac{\pi}{4n}}$$

For I_2, let $t = 1 - x$, then

$$I_2 = \int_0^{\frac{1}{2}} (1 - t + 2t^2 - t^3)^n \ln(3 - t)dt = \int_0^{\frac{1}{2}} (1 - x + 2x^2 - x^3)^n \ln(3 - x)dx$$

Take

$$a = 0, b = \frac{1}{2}, \varphi(x) = \ln(3 - x), h(x) = \ln(1 - x + 2x^2 - x^3), \xi = a = 0, h'(0) = -1$$

Therefore, by Theorem 3, we obtain

$$I_2 \sim \frac{\ln 3}{n} = o\left(\frac{1}{\sqrt{n}}\right)$$

Similarly, for

$$\int_0^1 f^n(x)dx = \int_0^{\frac{1}{2}} f^n(x)dx + \int_{\frac{1}{2}}^1 f^n(x)dx = I_3 + I_4$$

Thus,

$$\lim_{n\to\infty} \frac{\int_0^1 f^n(x)\ln(x+2)dx}{\int_0^1 f^n(x)dx} = \lim_{n\to\infty} \frac{\ln 2\sqrt{\frac{\pi}{4n}} + o\left(\frac{1}{\sqrt{n}}\right)}{\sqrt{\frac{\pi}{4n}} + o\left(\frac{1}{\sqrt{n}}\right)} = \boxed{\ln 2}$$

22 Riemann's Lemma

Theorem 1 If $f(x)$ is integrable on $[a, b]$ and $g(x)$ is a periodic function with period T and is integrable on $[0, T]$, then:

$$\lim_{n\to\infty} \int_a^b f(x)g(nx)dx = \frac{1}{T}\int_0^T g(x)dx \int_a^b f(x)dx$$

[**e.g. 22.1**] Calculate:

$$\lim_{n\to\infty} \int_0^1 \frac{x^2|\sin nx|}{1+x^2}dx$$

Solution: Let $f(x) = \frac{x^2}{1+x^2}$, $g(x) = |\sin x|$, and $T = \pi$. Then

$$\lim_{n\to\infty} \int_0^1 \frac{x^2|\sin nx|}{1+x^2}dx = \frac{1}{\pi}\int_0^\pi |\sin x|dx \int_0^1 \frac{x^2}{1+x^2}dx = \boxed{\frac{2}{\pi} - \frac{1}{2}}$$

[**e.g. 22.2**] Calculate:

$$\lim_{n\to\infty} \int_\pi^{2\pi} \frac{|\sin(nx) + \cos(nx)|}{x}dx$$

Solution: Let $f(x) = \frac{1}{x}$, $g(x) = |\sin x + \cos x|$, and $T = \pi$. Then

$$\lim_{n\to\infty} \int_\pi^{2\pi} \frac{|\sin(nx) + \cos(nx)|}{x}dx = \frac{1}{\pi}\int_\pi^{2\pi} \frac{1}{x}dx \int_0^\pi |\sin x + \cos x|dx = \boxed{\frac{2\sqrt{2}\ln 2}{\pi}}$$

[**e.g. 22.3**] Calculate:

$$\lim_{n\to\infty} \int_0^1 \frac{\sin^2(nx)}{1+x^2}dx$$

Solution: Let $f(x) = \frac{1}{1+x^2}$, $g(x) = \sin^2 x$, and $T = \pi$. Then

$$\lim_{n\to\infty} \int_0^1 \frac{\sin^2(nx)}{1+x^2}dx = \frac{1}{\pi}\int_0^1 \frac{1}{1+x^2}dx \int_0^\pi \sin^2 xdx = \boxed{\frac{\pi}{8}}$$

[e.g. 22.4] Given $\int_0^{\pi} \frac{\sin \frac{2n+1}{2}x}{2\sin \frac{x}{2}}dx = \frac{\pi}{2}$, prove Dirichlet's integral:

$$\lim_{n\to\infty} \int_0^{n} \frac{\sin x}{x}dx = \int_0^{+\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$$

Proof: Consider

$$\lim_{n\to\infty} \int_0^{\pi} \left(\frac{1}{2\sin \frac{x}{2}} - \frac{1}{x}\right) \sin\left(n+\frac{1}{2}\right)xdx = \lim_{n\to\infty} \int_0^{\pi} \left(\frac{1}{2\sin \frac{x}{2}} - \frac{1}{x}\right) \sin(nx)dx$$

Let $f(x) = \frac{1}{2\sin \frac{x}{2}} - \frac{1}{x}$, $g(x) = \sin x$, and $T = 2\pi$. Then

$$\lim_{n\to\infty} \int_0^{\pi} \left(\frac{1}{2\sin \frac{x}{2}} - \frac{1}{x}\right) \sin\left(n+\frac{1}{2}\right)xdx = \frac{1}{2\pi}\int_0^{2\pi} \sin xdx \int_0^{\pi} \left(\frac{1}{2\sin \frac{x}{2}} - \frac{1}{x}\right)dx = 0$$

Since $\int_0^{\pi} \frac{\sin \frac{2n+1}{2}x}{2\sin \frac{x}{2}}dx = \frac{\pi}{2}$, then

$$\lim_{n\to\infty} \int_0^{\pi} \frac{\sin(n+\frac{1}{2})x}{x}dx = \lim_{n\to\infty} \int_0^{\pi} \frac{\sin(nx)}{x}dx = \frac{\pi}{2}$$

Let $t = nx$ and $dx = \frac{1}{n}dt$. Thus

$$\lim_{n\to\infty} \int_0^{n} \frac{\sin x}{x}dx = \frac{\pi}{2}$$

That is

$$\lim_{n\to\infty} \int_0^{n} \frac{\sin x}{x}dx = \int_0^{+\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$$

23 Special Functions

In some limits involving integrals, especially when evaluating improper integrals, we may encounter several types of special improper integrals. If we are familiar with these conclusions, we can easily obtain the results.

23.1 The Γ Function

The Γ function (Gamma function) is the most fundamental and important function among special functions, and many special functions can be represented as combinations of it. Therefore, the Γ function has numerous applications.

Definition 1

$$\Gamma(z) = \int_0^{+\infty} x^{z-1}e^{-x}dx$$

Theorem 1 Recursion Formula:

$$\Gamma(x+1) = x\Gamma(x)$$

Complement Formula:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

Theorem 2 (Limit Definition)

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)}$$

Theorem 3 (Infinite Product Definition)

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty} \exp\left(\left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^{z} - 1\right)$$

$$\Gamma(z) = \frac{1}{z}e^{-\gamma z}\prod_{n=1}^{\infty} \exp\left(\left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}} - 1\right)$$

corollary 1

$$\Gamma(z) = \lim_{n\to\infty}\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n dt$$

[**e.g.23.1.1**] Calculate:

$$\lim_{n\to\infty} \frac{n!}{(a+1)(a+2)\cdots(a+n)} (a>1)$$

(Method 1) According to the limit definition of the Γ function, we know

$$\lim_{n\to\infty} \frac{n!}{(a+1)(a+2)\cdots(a+n)} = \lim_{n\to\infty} \frac{n^a n!}{a(a+1)(a+2)\cdots(a+n)} \cdot \lim_{n\to\infty}\frac{a}{n^a} = \Gamma(a)\cdot 0 = \boxed{0}$$

(Method 2) Since $a>1$, there must exist a positive integer b such that $1\le b\le a<b+1$, therefore:

$$\begin{aligned}\frac{n!}{(n+b+1)!}(b+1)! &= \frac{n!}{(b+2)(b+3)\cdots(b+n+1)}\\ &\le \frac{n!}{(a+1)(a+2)\cdots(a+n)}\\ &\le \frac{n!}{(b+1)(b+2)\cdots(b+n)} = \frac{n!}{(n+b)!}b!\end{aligned}$$

Since $b\ge 1$, then

$$0 \le \frac{n!}{(n+b+1)!}(b+1)! \le \frac{(b+1)!}{n+1}$$

$$0 \le \frac{n!}{(n+b)!}b! \le \frac{b!}{n+1}$$

Therefore,

$$\lim_{n\to\infty}\frac{n!}{(n+b+1)!}(b+1)! = \lim_{n\to\infty}\frac{n!}{(n+b)!}b! = 0$$

Hence,

$$\lim_{n\to\infty} \frac{n!}{(a+1)(a+2)\cdots(a+n)} = \boxed{0}$$

[e.g.23.1.2] Calculate:

$$\lim_{n\to\infty}\left(n-\Gamma\left(\frac{1}{n}\right)\right)$$

Solution:

$$\lim_{n\to\infty}\left(n-\Gamma\left(\frac{1}{n}\right)\right)=\lim_{n\to\infty}\frac{1-\frac{1}{n}\Gamma\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

Therefore, it suffices to calculate

$$\lim_{x\to 0^+}\frac{1-x\Gamma(x)}{x}$$

and then apply Heine's theorem to obtain the original limit. Since

$$\Gamma(x+1)=x\Gamma(x)$$

we have

$$\begin{aligned}
&\lim_{x\to 0^+}\frac{1-x\Gamma(x)}{x}\\
&=\lim_{x\to 0^+}\frac{1-\Gamma(x+1)}{x}\\
&=\lim_{x\to 0^+}\frac{-\Gamma'(x+1)}{1}\\
&=-\Gamma'(1)
\end{aligned}$$

Next, we find $\Gamma'(1)$. Since

$$\Gamma'(z)=\frac{d}{dz}\int_0^{+\infty}x^{z-1}e^{-x}dx=\int_0^{+\infty}\frac{\partial}{\partial z}x^{z-1}e^{-x}dx=\int_0^{+\infty}x^{z-1}e^{-x}\ln x dx$$

Let $z=1$, then

$$\Gamma'(1)=\int_0^{+\infty}e^{-x}\ln x dx=\underbrace{\int_0^1 e^{-x}\ln x dx}_{I_1}+\underbrace{\int_1^{+\infty}e^{-x}\ln x dx}_{I_2}$$

$$\begin{aligned}
I_1&=\int_0^1 e^{-x}\ln x dx\\
&=-\int_0^1\ln x d(e^{-x}-1)\\
&=-(e^{-x}-1)\ln x|_0^1+\int_0^1\frac{e^{-x}-1}{x}dx
\end{aligned}$$

And since

$$-(e^{-x}-1)\ln x|_0^1=\lim_{x\to 0}(e^{-x}-1)\ln x=\lim_{x\to 0}(-x)\ln x=0$$

we have $I_1=\int_0^1\frac{e^{-x}-1}{x}dx$. Similarly,

$$\begin{aligned}
I_2&=\int_1^{+\infty}e^{-x}\ln x dx\\
&=-\int_1^{+\infty}\ln x d(e^{-x})\\
&=-(e^{-x})\ln x|_1^{+\infty}+\int_1^{+\infty}\frac{e^{-x}}{x}dx
\end{aligned}$$

so $I_2 = \int_1^{+\infty} \frac{e^{-x}}{x} dx$. Then

$$\Gamma'(1) = I_1 + I_2 = \int_0^1 \frac{e^{-x} - 1}{x} dx + \int_1^{+\infty} \frac{e^{-x}}{x} dx$$

From Arzela's Dominated Convergence Theorem, and

$$\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}$$

we have

$$I_1 = \int_0^1 \frac{e^{-x} - 1}{x} dx = \int_0^1 \lim_{n\to\infty} \frac{\left(1 - \frac{x}{n}\right)^n - 1}{x} dx = \lim_{n\to\infty} \int_0^1 \frac{\left(1 - \frac{x}{n}\right)^n - 1}{x} dx$$

$$I_2 = \int_1^{+\infty} \frac{e^{-x}}{x} dx = \int_1^{+\infty} \lim_{n\to\infty} \frac{\left(1 - \frac{x}{n}\right)^n}{x} dx = \lim_{n\to\infty} \int_1^{n} \frac{\left(1 - \frac{x}{n}\right)^n}{x} dx$$

Therefore,

$$\begin{aligned}
&\Gamma'(1) \\
&= \lim_{n\to\infty} \left[\int_0^1 \frac{\left(1 - \frac{x}{n}\right)^n - 1}{x} dx + \int_1^n \frac{\left(1 - \frac{x}{n}\right)^n}{x} dx\right] \\
&= \lim_{n\to\infty} \left[\int_0^n \frac{\left(1 - \frac{x}{n}\right)^n - 1}{x} dx + \int_1^n \frac{1}{x} dx\right] \\
&= \lim_{n\to\infty} \left[\underbrace{\int_0^n \frac{\left(1 - \frac{x}{n}\right)^n - 1}{x} dx}_{I_3} + \ln n\right]
\end{aligned}$$

For I_3, let $t = 1 - \frac{x}{n}, dx = -ndt$. Therefore,

$$\begin{aligned}
I_3 &= \int_0^n \frac{\left(1 - \frac{x}{n}\right)^n - 1}{x} dx \\
&= -\int_0^1 \frac{1 - t^n}{1 - t} dt \\
&= -\int_0^1 (1 + t + t^2 + \cdots + t^{n-1}) dt \\
&= -\int_0^1 \sum_{k=1}^{n} t^{k-1} dt \\
&= -\sum_{k=1}^{n} \int_0^1 t^{k-1} dt \\
&= -\sum_{k=1}^{n} \frac{1}{k}
\end{aligned}$$

Then

$$\Gamma'(1) = \lim_{n\to\infty} \left(\ln n - \sum_{k=1}^{n} \frac{1}{k}\right) = -\gamma$$

Therefore,

$$\lim_{n\to\infty} \left(n - \Gamma\left(\frac{1}{n}\right)\right) = -\Gamma'(1) = \boxed{\gamma}$$

[e.g.23.1.3] Calculate:

$$\lim_{n\to\infty} \frac{\sqrt[n]{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n}\right)}}{n}$$

Solution:

$$\lim_{n\to\infty} \frac{\sqrt[n]{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n}\right)}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n}\right)}{n^n}}$$

We know that if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, then $\lim_{n\to\infty} \sqrt[n]{a_n} = l$.

Let

$$x_n = \frac{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n}\right)}{n^n}$$

Then

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\Gamma\left(\frac{1}{n+1}\right)}{n+1}\left(1-\frac{1}{n+1}\right)^n$$

$$\lim_{n\to\infty} \Gamma\left(\frac{1}{n+1}\right)\frac{1}{n+1} = \lim_{x\to 0} x\Gamma(x) = \lim_{x\to 0}\Gamma(x+1) = 1$$

Therefore

$$\lim_{n\to\infty} \frac{\sqrt[n]{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n}\right)}}{n} = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n = \boxed{\frac{1}{e}}$$

[e.g.23.1.4] Calculate:

$$\lim_{n\to\infty} \frac{\Gamma\left(\frac{1}{n+2}\right)}{\Gamma\left(\frac{1}{n+1}\right)}$$

Solution: By the Euler's reflection formula, we have:

$$\Gamma\left(\frac{1}{n+2}\right)\Gamma\left(1-\frac{1}{n+2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{n+2}\right)}$$

$$\Gamma\left(\frac{1}{n+1}\right)\Gamma\left(1-\frac{1}{n+1}\right) = \frac{\pi}{\sin\left(\frac{\pi}{n+1}\right)}$$

Then

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\Gamma\left(\frac{1}{n+2}\right)}{\Gamma\left(\frac{1}{n+1}\right)} \\
&= \lim_{n\to\infty} \frac{\Gamma\left(1-\frac{1}{n+1}\right)\sin\left(\frac{\pi}{n+1}\right)}{\Gamma\left(1-\frac{1}{n+2}\right)\sin\left(\frac{\pi}{n+2}\right)} \\
&= \lim_{n\to\infty} \frac{\Gamma\left(1-\frac{1}{n+1}\right)\left(\frac{\pi}{n+1}\right)}{\Gamma\left(1-\frac{1}{n+2}\right)\left(\frac{\pi}{n+2}\right)} \\
&= \lim_{n\to\infty} \frac{\Gamma\left(1-\frac{1}{n+1}\right)}{\Gamma\left(1-\frac{1}{n+2}\right)} \\
&= \frac{\Gamma(1)}{\Gamma(1)} \\
&= \boxed{1}
\end{aligned}$$

[e.g.23.1.5] Proof:

$$\lim_{n\to\infty}\left[\Gamma\left(\frac{1}{n+2}\right)-\Gamma\left(\frac{1}{n+1}\right)\right]=1$$

Proof: By the complement formula, we have

$$\Gamma\left(\frac{1}{n+2}\right)\Gamma\left(1-\frac{1}{n+2}\right)=\frac{\pi}{\sin\left(\frac{\pi}{n+2}\right)}$$

$$\Gamma\left(\frac{1}{n+1}\right)\Gamma\left(1-\frac{1}{n+1}\right)=\frac{\pi}{\sin\left(\frac{\pi}{n+1}\right)}$$

Therefore:

$$\begin{aligned}
&\lim_{n\to\infty}\left(\Gamma\left(\frac{1}{n+2}\right)-\Gamma\left(\frac{1}{n+1}\right)\right)\\
&=\lim_{n\to\infty}\pi\left(\frac{1}{\Gamma\left(1-\frac{1}{n+2}\right)\sin\left(\frac{\pi}{n+2}\right)}-\frac{1}{\Gamma\left(1-\frac{1}{n+1}\right)\sin\left(\frac{\pi}{n+1}\right)}\right)\\
&=\lim_{n\to\infty}\pi\left(\frac{\Gamma\left(1-\frac{1}{n+1}\right)\sin\left(\frac{\pi}{n+1}\right)-\Gamma\left(1-\frac{1}{n+2}\right)\sin\left(\frac{\pi}{n+2}\right)}{\Gamma\left(1-\frac{1}{n+2}\right)\sin\left(\frac{\pi}{n+2}\right)\cdot\Gamma\left(1-\frac{1}{n+1}\right)\sin\left(\frac{\pi}{n+1}\right)}\right)\\
&=\lim_{n\to\infty}\pi\left(\frac{\Gamma\left(1-\frac{1}{n+1}\right)\sin\left(\frac{\pi}{n+1}\right)-\Gamma\left(1-\frac{1}{n+2}\right)\sin\left(\frac{\pi}{n+2}\right)}{\left(\frac{\pi}{n+2}\right)\cdot\left(\frac{\pi}{n+1}\right)}\right)\\
&=\lim_{n\to\infty}\frac{n^2}{\pi}\cdot\left(\pi\frac{\cos\left(\frac{\pi}{\xi}\right)\Gamma\left(1-\frac{1}{\xi}\right)}{\xi^2}-\frac{\sin\frac{\pi}{\xi}\Gamma'\left(1-\frac{1}{\xi}\right)}{\xi^2}\right)\quad(n+1<\xi<n+2)\\
&=\lim_{n\to\infty}\frac{n^2}{\xi^2}\cos\left(\frac{\pi}{\xi}\right)\Gamma\left(1-\frac{1}{\xi}\right)\\
&=1
\end{aligned}$$

[e.g.23.1.6] Calculate:

$$\lim_{n\to\infty}\left[\sqrt[n+1]{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n+1}\right)}-\sqrt[n]{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n}\right)}\right]$$

Solution: Let

$$a_n=\sqrt[n]{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n}\right)}\sim\frac{n}{e}(n\to\infty)$$

Then

$$\begin{aligned}
&\lim_{n\to\infty}\left(\sqrt[n+1]{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n+1}\right)}-\sqrt[n]{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n}\right)}\right)\\
&=\lim_{n\to\infty}\left(e^{\ln a_{n+1}}-e^{\ln a_n}\right)\\
&=\lim_{n\to\infty}a_n\left(e^{\ln a_{n+1}-\ln a_n}-1\right)\\
&=\lim_{n\to\infty}\frac{n}{e}\left(e^{\ln\frac{a_{n+1}}{a_n}-1}\right)\\
&=\frac{1}{e}\lim_{n\to\infty}\frac{\ln a_{n+1}-\ln a_n}{\frac{1}{n}}\\
&=\frac{1}{e}\lim_{n\to\infty}\frac{\frac{1}{n+1}\sum_{k=1}^{n+1}\ln\Gamma\left(\frac{1}{k}\right)-\frac{1}{n}\sum_{k=1}^{n}\ln\Gamma\left(\frac{1}{k}\right)}{\frac{1}{n}}
\end{aligned}$$

Combine the numerator:

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\frac{1}{n+1}\sum_{k=1}^{n+1}\ln\Gamma\left(\frac{1}{k}\right)-\frac{1}{n}\sum_{k=1}^{n}\ln\Gamma\left(\frac{1}{k}\right)}{\frac{1}{n}}\\
&=\lim_{n\to\infty}\frac{-\frac{1}{n(n+1)}\sum_{k=1}^{n}\ln\Gamma\left(\frac{1}{k}\right)+\ln\Gamma\left(\frac{1}{n+1}\right)}{\frac{1}{n}}\\
&=\lim_{n\to\infty}\frac{-\sum_{k=1}^{n}\ln\Gamma\left(\frac{1}{k}\right)+n\ln\Gamma\left(\frac{1}{n+1}\right)}{n+1}
\end{aligned}$$

Using Stolz's theorem, we get

$$\begin{aligned}
&\lim_{n\to\infty}\frac{-\sum_{k=1}^{n}\ln\Gamma\left(\frac{1}{k}\right)+n\ln\Gamma\left(\frac{1}{n+1}\right)}{n+1}\\
&=\lim_{n\to\infty}\frac{-\ln\Gamma\left(\frac{1}{n+1}\right)+(n+1)\ln\Gamma\left(\frac{1}{n+2}\right)-n\ln\Gamma\left(\frac{1}{n+1}\right)}{1}\\
&=\lim_{n\to\infty}(n+1)\ln\left(\frac{\Gamma\left(\frac{1}{n+2}\right)}{\Gamma\left(\frac{1}{n+1}\right)}\right)\\
&=\lim_{n\to\infty}(n+1)\ln\left(\frac{\Gamma\left(\frac{1}{n+2}\right)}{\Gamma\left(\frac{1}{n+1}\right)}-1+1\right)
\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{\Gamma\left(\frac{1}{n+2}\right)}{\Gamma\left(\frac{1}{n+1}\right)}=1$$

Using equivalent infinitesimals, we get:

$$\begin{aligned}
&\lim_{n\to\infty}(n+1)\ln\left(\frac{\Gamma\left(\frac{1}{n+2}\right)}{\Gamma\left(\frac{1}{n+1}\right)}-1+1\right)\\
&=\lim_{n\to\infty}(n+1)\frac{\Gamma\left(\frac{1}{n+2}\right)-\Gamma\left(\frac{1}{n+1}\right)}{\Gamma\left(\frac{1}{n+1}\right)}\\
&=\frac{\lim\limits_{n\to\infty}\left[\Gamma\left(\frac{1}{n+2}\right)-\Gamma\left(\frac{1}{n+1}\right)\right]}{\lim\limits_{n\to\infty}\frac{1}{n+1}\Gamma\left(\frac{1}{n+1}\right)}\\
&=1
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{n\to\infty}\left[\sqrt[n+1]{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n+1}\right)}-\sqrt[n]{\Gamma\left(\frac{1}{1}\right)\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{n}\right)}\right]\\
&=\frac{1}{e}\lim_{n\to\infty}\frac{\frac{1}{n+1}\sum_{k=1}^{n+1}\ln\Gamma\left(\frac{1}{k}\right)-\frac{1}{n}\sum_{k=1}^{n}\ln\Gamma\left(\frac{1}{k}\right)}{\frac{1}{n}}\\
&=\boxed{\frac{1}{e}}
\end{aligned}$$

23.2 B Function

The B function (Beta function) is another special and very important function, which can be expressed using the Γ function.

Theorem 4

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx \quad (p,q>0)$$

Theorem 5

$$B(p,q)=B(q,p)$$

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

corollary 2

$$B(x,1-x)=\frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)}=\frac{\pi}{\sin\pi x}$$

corollary 3

$$\int_0^{\frac{\pi}{2}} \sin^{a-1} x \cdot \cos^{b-1} x dx = \frac{1}{2} B\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{a}{2}\right) \cdot \Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}$$

Proof: Let $t = \sin x$ then $x = \arcsin t$ and $dx = \frac{dt}{\sqrt{1-t^2}}$. Therefore,

$$\int_0^{\frac{\pi}{2}} \sin^{a-1} x \cdot \cos^{b-1} x dx = \int_0^1 t^{a-1} \cdot (1-t^2)^{\frac{b}{2}-1} dt = \frac{1}{2} \frac{\Gamma\left(\frac{a}{2}\right) \cdot \Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}$$

[**e.g.23.2.1**] Calculate:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

where $a_n = \int_0^1 x(1-x^3)^n dx$

Solution: (Method 1): Make the substitution $t = x^3$ and $dx = \frac{1}{3} t^{-\frac{2}{3}} dt$.
Therefore:

$$\begin{aligned}
&\int_0^1 x(1-x^3)^n dx \\
&= \frac{1}{3} \int_0^1 t^{-\frac{1}{3}} (1-t)^n dt \\
&= \frac{1}{3} B\left(\frac{2}{3}, n+1\right) = \frac{1}{3} \frac{\Gamma\left(\frac{2}{3}\right) \Gamma(n+1)}{\Gamma\left(n+\frac{5}{3}\right)} \\
&= \frac{1}{3} \frac{\Gamma\left(\frac{2}{3}\right) n!}{(n+\frac{2}{3})(n-\frac{1}{3}) \cdots \frac{2}{3} \Gamma\left(\frac{2}{3}\right)} \\
&= \frac{3^n n!}{(3n+2)(3n-1)\cdots 2}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{3^{n+1}(n+1)!}{(3n+5)(3n+2)\cdots 2} \cdot \frac{(3n+2)(3n-1)\cdots 2}{3^n n!} \\
&= \lim_{n\to\infty} \frac{3n+3}{3n+5} \\
&= \boxed{1}
\end{aligned}$$

(Method 2): Repeatedly apply integration by parts:

$$
\begin{aligned}
a_n &= \int_0^1 x(1-x^3)^n dx \\
&= \frac{1}{2}\int_0^1 (1-x^3)^n d(x^2) \\
&= \frac{3n}{2}\int_0^1 x^4(1-x^3)^{n-1}dx \\
&= \frac{3n}{2}\cdot\frac{1}{5}\int_0^1 (1-x^3)^{n-1}d(x^5) \\
&= \frac{3n}{2}\cdot\frac{3(n-1)}{5}\int_0^1 x^7(1-x^3)^{n-2}dx \\
&= \cdots \\
&= \frac{3n}{2}\cdot\frac{3(n-1)}{5}\cdots\frac{3}{3n-1}\int_0^1 x^{3n+1}dx \\
&= \frac{3n}{2}\cdot\frac{3(n-1)}{5}\cdots\frac{3}{3n-1}\frac{1}{3n+2} \\
&= \frac{3^n n!}{(3n+2)(3n-1)\cdots 2}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty}\frac{3^{n+1}(n+1)!}{(3n+5)(3n+2)\cdots 2}\cdot\frac{(3n+2)(3n-1)\cdots 2}{3^n n!} \\
&= \lim_{n\to\infty}\frac{3n+3}{3n+5} \\
&= \boxed{1}
\end{aligned}
$$

[e.g.23.2.2] Calculate:

$$I_n = \lim_{x\to+\infty}\left(\ln^n x - n\int_0^x \frac{\ln^{n-1} t}{\sqrt{1+t^2}}dt\right)$$

Solution:

$$
\begin{aligned}
I_n &= \lim_{x\to+\infty}\left(\ln^n x - n\int_0^x \frac{\ln^{n-1} t}{\sqrt{1+t^2}}dt\right) \\
&= \lim_{x\to+\infty}\left(\ln^n x - \int_0^x \frac{t}{\sqrt{1+t^2}}d(\ln^n t)\right) \\
&= \lim_{x\to+\infty}\left(\ln^n x - \frac{x\ln^n x}{\sqrt{1+x^2}} + \int_0^x \frac{\ln^n t}{(1+t^2)^{\frac{3}{2}}}dt\right)
\end{aligned}
$$

$$\begin{aligned}
&= \lim_{x\to+\infty} \frac{(\sqrt{1+x^2}-x)\ln^n x}{\sqrt{1+x^2}} + \lim_{x\to+\infty}\int_0^x \frac{\ln^n t}{(1+t^2)^{\frac{3}{2}}}dt \\
&= \lim_{x\to+\infty} \frac{\ln^n x}{(\sqrt{1+x^2}+x)\sqrt{1+x^2}} + \int_0^{+\infty} \frac{\ln^n t}{(1+t^2)^{\frac{3}{2}}}dt \\
&= \int_0^{+\infty} \frac{\ln^n t}{(1+t^2)^{\frac{3}{2}}}dt
\end{aligned}$$

Considering the Beta function:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

Let $x = \frac{t^2}{1+t^2}(t \geq 0)$, then $dx = \frac{2t}{(1+t^2)^2}dt$. Thus,

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx = \int_0^{+\infty}\left(\frac{t^2}{1+t^2}\right)^{p-1}\left(\frac{1}{1+t^2}\right)^{q-1}\frac{2t}{(1+t^2)^2}dt = 2\int_0^{+\infty}\frac{t^{2p-1}}{(1+t^2)^{p+q}}dt$$

Let $p+q=\frac{3}{2}$, then

$$B\left(p,\frac{3}{2}-p\right) = 2\int_0^{+\infty}\frac{t^{2p-1}}{(1+t^2)^{\frac{3}{2}}}dt$$

Taking the n-th derivative of both sides with respect to p, we have

$$B_n = B^{(n)}\left(p,\frac{3}{2}-p\right) = 2^{n+1}\int_0^{+\infty}\frac{t^{2p-1}\ln^n t}{(1+t^2)^{\frac{3}{2}}}dt$$

Therefore,

$$\int_0^{+\infty}\frac{\ln^n t}{(1+t^2)^{\frac{3}{2}}}dt = \lim_{p\to\frac{1}{2}}\int_0^{+\infty}\frac{t^{2p-1}}{(1+t^2)^{\frac{3}{2}}}dt = \lim_{p\to\frac{1}{2}}\frac{B_n}{2^{n+1}} = \frac{B^{(n)}\left(p,\frac{3}{2}-p\right)}{2^{n+1}}$$

23.3 The ψ Function

The ψ function (DiGamma function) is the derivative of $\ln\Gamma(x)$, which is also a very important function closely related to the harmonic series.

Definition 2

$$\psi(x) = (\ln\Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}$$

According to the infinite product definition of the Γ function, $\Gamma(z) = \frac{1}{z}e^{-\gamma z}\prod_{n=1}^{\infty}\exp\left(\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}}\right)$, then

$$\ln\Gamma(x) = -\gamma x - \ln x + \sum_{k=1}^{\infty}\left(\frac{x}{k} - \ln\left(1+\frac{x}{k}\right)\right) = -\gamma x - \ln x + \sum_{k=1}^{\infty}\left(\frac{x}{k} - \ln(k+x) + \ln x\right)$$

Taking the derivative of both sides, we get

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+x}\right)$$

Thus:

corollary 4

$$\psi(x) = -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+x}\right)$$

Substituting $x = 1$, we have $\psi(1) = \frac{\Gamma'(1)}{\Gamma(1)} = \Gamma'(1) = -\gamma$

corollary 5

$$\psi(x) = -\gamma + \int_0^1 \frac{1-t^{x-1}}{1-t}dt$$

Proof:

$$\begin{aligned}
\psi(x) &= -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+x}\right) \\
&= -\gamma + \sum_{k=0}^{\infty}\left(\frac{t^{k+1}}{k+1} - \frac{t^{k+x}}{k+x}\right)\Bigg|_0^1 \\
&= -\gamma + \sum_{k=0}^{\infty}\int_0^1 (t^k - t^{k+x-1})dt \\
&= -\gamma + \sum_{k=0}^{\infty}\int_0^1 t^k(1-t^{x-1})dt \\
&= -\gamma + \int_0^1 \sum_{k=0}^{\infty} t^k(1-t^{x-1})dt \\
&= -\gamma + \int_0^1 \frac{1-t^{x-1}}{1-t}dt
\end{aligned}$$

corollary 6

$$\psi(x+n) - \psi(x) = \sum_{k=0}^{n-1}\frac{1}{x+k}$$

[e.g.23.3.1] Given $\psi(\frac{1}{3}) = -\gamma - \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\ln 3$ and $\psi(\frac{2}{3}) = -\gamma + \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\ln 3$, calculate:

$$\sum_{k=0}^{\infty}\frac{1}{(3k+1)(3k+2)(3k+3)}$$

Solution:

$$\sum_{k=0}^{\infty}\frac{1}{(3k+1)(3k+2)(3k+3)} = \frac{1}{6(k+\frac{1}{3})} - \frac{1}{3(k+\frac{2}{3})} + \frac{1}{6(k+1)}$$

And

$$\sum_{k=0}^{n-1}\frac{1}{x+k} = \psi(x+n) - \psi(x)$$

Therefore

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{1}{(3k+1)(3k+2)(3k+3)}\\&=\lim_{n\to\infty}\left(\frac{1}{6(k+\frac{1}{3})}-\frac{1}{3(k+\frac{2}{3})}+\frac{1}{6(k+1)}\right)\\&=\lim_{n\to\infty}\left(\frac{\psi(n)-\psi(\frac{1}{3})}{6}-\frac{\psi(n)-\psi(\frac{2}{3})}{3}+\frac{\psi(n)-\psi(1)}{6}\right)\\&=\frac{-\psi(\frac{1}{3})+2\psi(\frac{2}{3})-\psi(1)}{6}\\&=\boxed{\frac{\pi}{4\sqrt{3}}-\frac{3}{4}\ln 3}\end{aligned}$$

[e.g.23.3.2] Calculate:

$$\lim_{n\to\infty} n\left(\frac{1}{2}-(n-1)\left(\int_0^1\frac{x^n}{1+x^2}dx\right)\right)$$

Solution: Let $y=x^4$, then

$$\begin{aligned}&\int_0^1\frac{x^n}{1+x^2}dx=\frac{1}{4}\int_0^1\frac{y^{\frac{n-3}{4}}}{1+\sqrt{y}}dy\\&=\frac{1}{4}\int_0^1\frac{y^{\frac{n-3}{4}}-y^{\frac{n-1}{4}}}{1-y}dy\\&=\frac{1}{4}\left(\int_0^1\left(\frac{1-y^{\frac{n-3}{4}}}{1-y}-\frac{1-y^{\frac{n-1}{4}}}{1-y}\right)dy\right)\\&=\frac{1}{4}\left(\psi\left(\frac{n+3}{4}\right)-\psi\left(\frac{n+1}{4}\right)\right)\end{aligned}$$

Let $t=1-x, q=t-\frac{t^2}{t}$, then

$$\begin{aligned}&\frac{1}{1+x^2}=\frac{1}{t^2-2t+2}\\&=\frac{1}{2}\cdot\frac{1}{1-\left(t-\frac{t^2}{t}\right)}\\&=\frac{1}{2}\cdot\frac{1}{1-q}\\&=\frac{1}{2}\left(1+q+q^2+\cdots\right)\\&=\frac{1}{2}\left(1+\left((1-x)-\frac{(1-x)^2}{2}\right)+\left((1-x)-\frac{(1-x)^2}{2}\right)^2+\cdots\right)\end{aligned}$$

Therefore:

$$\int_0^1\frac{x^n}{1+x^2}dx=\frac{1}{2}\int_0^1 x^n\left(1+\left((1-x)-\frac{(1-x)^2}{2}\right)+\left((1-x)-\frac{(1-x)^2}{2}\right)^2+\cdots\right)dx$$

Considering the Beta function

$$\int_0^1 x^n(1-x)^m = B(n+1,m+1) = \frac{n!m!}{(m+n+1)!}$$

Therefore

$$\int_0^1 \frac{x^n}{1+x^2}dx = \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots\right)$$

Therefore

$$\begin{aligned}
&\lim_{n\to\infty} n\left(\frac{1}{2} - (n-1)\left(\int_0^1 \frac{x^n}{1+x^2}dx\right)\right) \\
&= \lim_{n\to\infty} n\left(\frac{1}{2} - (n-1)\left(\frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots\right)\right)\right) \\
&= \frac{1}{2}\lim_{n\to\infty}\left(\frac{2n}{n+1} - \frac{n(n-1)}{(n+2)(n+1)} + \cdots\right) \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

23.4 The ζ Function

Definition 3

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

corollary 7

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{t^{s-1}}{e^t-1}dt$$

[e.g.23.4.1] Let $s \geq 0$, find $\varphi(1), \varphi(2)$:

where

$$\varphi(s) = \int_0^{+\infty}\frac{\ln(1+sx^2)}{x(1+x^2)}dx$$

Let $t = x^2, dt = 2xdx$ then

$$\int_0^{+\infty}\frac{\ln(1+sx^2)}{x(1+x^2)}dx = \frac{1}{2}\int_0^{+\infty}\frac{\ln(1+st)}{t(1+t)}dt$$

For $s = 1$

$$\varphi(1) = \frac{1}{2}\int_0^{+\infty}\frac{\ln(1+t)}{t(1+t)}dt$$

Consider

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{t^{s-1}}{e^t-1}dt$$

Let $t = \ln(1+x)$ then

$$\zeta(2) = \frac{1}{\Gamma(2)}\int_0^{+\infty}\frac{\ln(1+x)}{x(1+x)}dx = \int_0^{+\infty}\frac{\ln(1+t)}{t(1+t)}dt = 2\varphi(1) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Thus

$$\varphi(1)=\boxed{\frac{\pi^2}{12}}$$

For $s=2$

$$\begin{aligned}
\varphi(2)&=\frac{1}{2}\int_0^{+\infty}\frac{\ln(1+2t)}{t(1+t)}dt\\
&=\int_0^{+\infty}\frac{\ln(1+t)}{t(2+t)}dt\quad(2t\to t)\\
&=\int_0^{+\infty}\frac{\ln(1+t)}{t(1+t)}\left(\frac{t+1}{t+2}\right)dt\\
&=\int_0^{+\infty}\frac{\ln(1+t)}{t(1+t)}\left(1-\frac{1}{t+2}\right)dt\\
&=\int_0^{+\infty}\frac{\ln(1+t)}{t(1+t)}dt-\int_0^{+\infty}\frac{\ln(1+t)}{t(1+t)(t+2)}dt\\
&=\int_0^{+\infty}\frac{\ln(1+t)}{t(1+t)}dt-\frac{1}{2}\left[\int_0^{+\infty}\frac{\ln(1+t)}{t(1+t)}dt-\int_0^{+\infty}\frac{\ln(1+t)}{(t+2)(1+t)}dt\right]\\
&=\frac{1}{2}\int_0^{+\infty}\frac{\ln(1+t)}{t(1+t)}dt+\frac{1}{2}\int_0^{+\infty}\frac{\ln(1+t)}{(1+t)(t+2)}dt
\end{aligned}$$

And

$$\varphi(1)=\frac{1}{2}\int_0^{+\infty}\frac{\ln(1+t)}{t(1+t)}dt=\int_0^1\frac{\ln(1+t)}{t}dt=\frac{\pi^2}{12}$$

Therefore, only $\int_0^{+\infty}\frac{\ln(1+t)}{(1+t)(t+2)}dt$ needs to be calculated, and

$$\begin{aligned}
&\int_0^{+\infty}\frac{\ln(1+t)}{(1+t)(t+2)}dt\\
&=\int_0^{+\infty}\frac{\ln(1+t)}{(1+t)}dt-\int_0^{+\infty}\frac{\ln(1+t)}{(t+2)}dt\\
&=\int_0^{+\infty}\frac{\ln(1+t)}{(1+t)}dt-\int_1^{+\infty}\frac{\ln(t)}{(t+1)}dt\\
&=\int_0^{1}\frac{\ln(1+t)}{(1+t)}dt+\int_1^{+\infty}\frac{\ln(1+\frac{1}{t})}{(t+1)}dt
\end{aligned}$$

And

$$\int_1^{+\infty}\frac{\ln(1+\frac{1}{t})}{(t+1)}dt=\int_0^1\frac{\ln(1+x)}{x(x+1)}dx\ (x=\frac{1}{t})=\int_0^1\frac{\ln(1+t)}{t(t+1)}dt$$

Then

$$\begin{aligned}
&\int_0^{1}\frac{\ln(1+t)}{(1+t)}dt+\int_1^{+\infty}\frac{\ln(1+\frac{1}{t})}{(t+1)}dt\\
&=\int_0^{1}\frac{\ln(1+t)}{(1+t)}dt+\int_0^{1}\frac{\ln(1+t)}{t(t+1)}dt\\
&=\int_0^{1}\frac{(t+1)\ln(1+t)}{t(t+1)}dt\\
&=\int_0^{1}\frac{\ln(1+t)}{t}dt=\frac{\pi^2}{12}
\end{aligned}$$

Thus

$$\varphi(2) = \frac{\pi^2}{12} + \frac{1}{2} \cdot \frac{\pi^2}{12} = \boxed{\frac{\pi^2}{8}}$$

24 Fourier Series

A periodic function that satisfies certain conditions can be expanded into a Fourier series, which is very useful in finding the convergence values of infinite series.

Theorem 1 (Dirichlet's Conditions) Let $f(x)$ be periodic with period $2l$ and satisfy the following conditions in $[-l, l]$:
(1) It is continuous or has only a finite number of discontinuities of the first kind;
(2) It has only a finite number of extreme points.
Then $f(x)$ can be expanded into a Fourier series

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right)$$

where

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

denoted as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right)$$

Then when x is a point of continuity of $f(x)$, $S(x) = f(x)$; when x is a point of discontinuity of $f(x)$,

$$S(x) = \frac{f(x+0) + f(x-0)}{2}$$

corollary 1 Let $f(x)$ be periodic with period $2l$ and satisfy Dirichlet's conditions, then:
(1) When $f(x)$ is an even function,

$$a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx, b_n = 0$$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

(2) When $f(x)$ is an odd function,

$$a_n = 0, b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

For simple power functions $f(x) = x^a$ (where $a = 2k, k \in N$), by expanding them into Fourier series through recursion, we can obtain the convergence values of infinite series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^a}$ and $\sum_{n=1}^{\infty} \frac{1}{n^a}$.

[e.g.24.1] Find the convergence values of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution: Expand $f(x) = x^2$ into a Fourier series on $[-\pi, \pi]$, then:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx = 4\frac{(-1)^n}{n^2}, a_0 = \frac{2}{3}\pi^2, b_n = 0$$

Therefore

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2} \cos(nx)$$

Let $x = 0$, then

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2} = 0$$

Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \boxed{-\frac{\pi^2}{12}}$$

Let $x = \pi$, then

$$-\frac{2\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{1}{n^2} = 0$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \boxed{\frac{\pi^2}{6}}$$

[e.g.24.2] Find $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$

Solution: Expand $f(x) = x$ into a Fourier series on $[-\pi, \pi]$, then:

$$a_n = 0, b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = 2\frac{(-1)^{n-1}}{n}$$

Therefore

$$x = \sum_{n=1}^{\infty} 2\frac{(-1)^{n-1}}{n} \sin(nx)$$

Let $x = \frac{\pi}{2}$, then

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} 2\frac{(-1)^{n-1}}{n} \sin(nx) = \sum_{n=1}^{\infty} \left(2\frac{(-1)^{2n-1}}{2n} \sin(n\pi) + 2\frac{(-1)^{2n-2}}{2n-1} \sin\left(n\pi - \frac{\pi}{2}\right)\right) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$$

Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \boxed{\frac{\pi}{4}}$$

25 Bernoulli Numbers

Definition 1 (Recursive Definition) Let

$$\delta_{m,0} = \begin{cases} 1, & \text{if } m = 0 \\ 0, & \text{if } m \neq 0 \end{cases}$$

Then

$$B_m = \delta_{m,0} - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k}{m-k+1}$$

Definition 2 (Generating Function) For the sequence $\{a_n\}$, denote

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

as the generating function of the sequence $\{a_n\}$.

Definition 3 The Bernoulli numbers are defined by the generating function $\frac{x}{e^x-1}$, where

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

Thus, by performing the Taylor expansion of $\frac{x}{e^x-1}$, we can obtain B_n:

$$B_n = \lim_{x \to 0} \frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right)$$

Definition 4 (Bernoulli Polynomials) The polynomials of the following form are called Bernoulli polynomials:

$$B_k(n) = \sum_{i=0}^{k} \binom{k}{i} B_i n^{k-i}$$

(Power Sum Formula) With Bernoulli polynomials, we can compute

$$\sum_{k=1}^{n} k^p = \frac{B_{p+1}(n) - B_{p+1}}{p+1} = \frac{n^{p+1}}{p+1} + \frac{1}{2} n^p + \sum_{k=1}^{p-1} \frac{p(p-1)\cdots(p-k+1)}{(k+1)!} B_{k+1} n^{p-k}$$

[**e.g.25.1**] Calculate:

$$\lim_{n \to \infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} \quad (k \in \mathbb{N})$$

Solution:

$$\begin{aligned} & \lim_{n \to \infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} \\ = & \lim_{n \to \infty} \frac{\frac{n^{k+1}}{k+1} + \frac{1}{2} n^k + o(n^k)}{n^{k+1}} \\ = & \boxed{\frac{1}{k+1}} \end{aligned}$$

[e.g.25.2] Calculate:

$$\lim_{n\to\infty}\left(\frac{1^k+2^k+\cdots+n^k}{n^k}-\frac{n}{k+1}\right)\quad (k\in\mathbb{N})$$

Solution:

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1^k+2^k+\cdots+n^k}{n^k}-\frac{n}{k+1}\right)\\
=&\lim_{n\to\infty}\left(\frac{\frac{n^{k+1}}{k+1}+\frac{1}{2}n^k+o(n^k)}{n^k}-\frac{n}{k+1}\right)\\
=&\lim_{n\to\infty}\left(\frac{1}{2}+\frac{o(n^k)}{n^k}\right)\\
=&\boxed{\frac{1}{2}}
\end{aligned}$$

26 Euler-Maclaurin Summation Formula

The Euler-Maclaurin summation formula, abbreviated as the EM formula, connects summation and integration.

Theorem 1

$$\begin{aligned}
\sum_{k=1}^{n} f(k) &= \int_0^n f(x)dx + \frac{f(n)-f(0)}{2} + \int_0^n \psi(x)f'(x)dx\\
&= \int_1^n f(x)dx + \frac{f(n)+f(1)}{2} + \int_1^n \psi(x)f'(x)dx
\end{aligned}$$

where $\psi(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

Theorem 2 (General Form of the EM Formula) Let the function $f(x)$ have continuous derivatives up to the $2n-1$st order on the interval $[a,b]$. The Euler-Maclaurin formula is given by:

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \frac{f(a)+f(b)}{2} + \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}\int_a^b f^{(2n-1)}(x)dx\cdot\operatorname{sgn}(b-a)$$

where B_n are the Bernoulli numbers and $\operatorname{sgn}(b-a)$ denotes the sign of $b-a$ (note: the original expression $\left[f^{(2n-1)}(b)-f^{(2n-1)}(a)\right]dx$ is adjusted here for clarity and correctness, assuming the intended meaning is the integral difference multiplied by the sign, though typically this term is presented differently in standard Euler-Maclaurin expansions).

[e.g.26.1] Prove:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\cos\sqrt{k}=0$$

Proof:

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\cos\sqrt{k}\\
&=\frac{\cos 1+\cos\sqrt{n}}{n}+\frac{1}{n}\int_1^n\cos\sqrt{x}dx+\frac{1}{n}\int_1^n\psi(x)f'(x)dx\\
&=\frac{\cos 1+\cos\sqrt{n}}{n}+\frac{2\cos\sqrt{n}+2\sqrt{n}\sin\sqrt{n}-2\sin 1-2\cos 1}{n}+\frac{1}{n}\int_1^n\psi(x)f'(x)dx
\end{aligned}$$

The limits of the first two terms are both 0, so only the limit of the last term needs to be considered. $f'(x)=\frac{\sin\sqrt{x}}{2\sqrt{x}},\psi(x)=x-\lfloor x\rfloor-\frac{1}{2}$ Therefore,

$$\left|\frac{1}{n}\int_1^n\psi(x)f'(x)dx\right|\leq\frac{1}{n}\left|\int_1^n\frac{\sin\sqrt{x}}{2\sqrt{x}}dx\right|\leq\frac{1}{n}\left|\int_1^n\frac{1}{2\sqrt{x}}dx\right|=\frac{\sqrt{n}-1}{n}\to 0\quad(n\to\infty)$$

Thus,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\cos\sqrt{k}=0$$

27 The Boundary Addition Problem

The essence of the boundary addition problem in sequence limits is to find the coefficients of each term in the asymptotic expansion of the sequence a_n. In layman's terms, if $\lim_{n\to\infty}a_n=a$, then a_n can be expressed as:

$$a_n=a+\frac{A_1}{n}+\frac{A_2}{n^2}+\cdots+\frac{A_k}{n^k}+o\left(\frac{1}{n^k}\right)$$

Therefore, we may encounter problems like finding $\lim_{n\to\infty}n(a_n-a)$, $\lim_{n\to\infty}n\left[n(a_n-a)-A_1\right]$, or even

$$\lim_{n\to\infty}n\left\{\cdots n\left[n\left[n\left[n(a_n-a)-A_1\right]-A_2\right]\cdots-A_k\right]\right\}$$

There are many solutions to this problem, such as using high-order EM formulas or Taylor formulas for algebraic simplification calculations, or using Stolz's theorem or other methods.

Before that, we first prove a conclusion mentioned in Definition of Definite Integral.

[e.g.27.1] If $f(x)$ has continuous $p+1$ derivatives on $[a,b]$, prove that:

$$\lim_{n\to\infty}n^p\left\{\left[\sum_{k=0}^{p-1}(-1)^k\cdot\frac{(b-a)^{k+1}}{k!\cdot(k+1)n^{k+1}}\sum_{i=1}^{n}f^{(k)}(x_i)\right]-\int_a^b f(x)dx\right\}=\frac{(-1)^{p+1}(b-a)^p}{(p+1)!}\left[f^{(p-1)}(b)-f^{(p-1)}(a)\right]$$

where $x_i=a+i\frac{b-a}{n}$ for $i=1,2,...,n$.

Proof: Insert $n-1$ points into the interval $[a,b]$:

$$x_i=a+i\frac{b-a}{n}\quad(i=1,2,...,n)$$

Let I_i denote each subinterval:

$$I_i=\left[a+(i-1)\frac{b-a}{n},a+i\frac{b-a}{n}\right)\quad(i=1,2,...,n)$$

Expand $f(x)$ at x_i using Taylor's formula with the Lagrange remainder term up to the pth order. Then there exists

$$\xi_i \in \left(a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}\right) \quad (i = 1, 2, ..., n)$$

such that:

$$f(x) = \sum_{k=0}^{p} \frac{f^{(k)}(x_i)}{k!}(x - x_i)^k + \frac{f^{(p+1)}(\xi_i)}{(p+1)!}(x - x_i)^{p+1}$$

Since $f(x)$ has continuous $p+1$ derivatives on $[a, b]$, then $\exists M = \max|f^{(p+1)}([a, b])|$.

Notice that:

$$\left[\sum_{k=0}^{p-1} (-1)^k \cdot \frac{(b-a)^{k+1}}{k! \cdot (k+1)n^{k+1}} \sum_{i=1}^{n} f^{(k)}(x_i)\right] = \sum_{i=1}^{n} n^p \int_{a+(i-1)\frac{b-a}{n}}^{a+i\frac{b-a}{n}} \sum_{k=0}^{p-1} \frac{f^{(k)}(x_i)}{k!}(x - x_i)^k dx$$

Consider

$$\left|n^p \left\{\left[\sum_{k=0}^{p-1} (-1)^k \cdot \frac{(b-a)^{k+1}}{k! \cdot (k+1)n^{k+1}} \sum_{i=1}^{n} f^{(k)}(x_i)\right] - \int_a^b f(x)dx\right\} + \sum_{i=1}^{n} n^p \int_{a+(i-1)\frac{b-a}{n}}^{a+i\frac{b-a}{n}} \frac{f^{(p)}(x_i)}{p!}(x - x_i)^p dx\right|$$
$$= \left|\sum_{i=1}^{n} n^p \left[\int_{a+(i-1)\frac{b-a}{n}}^{a+i\frac{b-a}{n}} \sum_{k=0}^{p} \frac{f^{(k)}(x_i)}{k!}(x - x_i)^k - f(x)dx\right]\right|$$
$$= \left|\sum_{i=1}^{n} n^p \int_{a+(i-1)\frac{b-a}{n}}^{a+i\frac{b-a}{n}} \frac{f^{(p+1)}(\xi_i)}{(p+1)!}(x - x_i)^{p+1} dx\right|$$

And since $|f^{(p+1)}(\xi_i)| \leq M$, then

$$\left|\sum_{i=1}^{n} n^p \int_{a+(i-1)\frac{b-a}{n}}^{a+i\frac{b-a}{n}} \frac{f^{(p+1)}(\xi_i)}{(p+1)!}(x - x_i)^{p+1} dx\right|$$
$$\leq \frac{M}{(p+1)!}\left|\sum_{i=1}^{n} n^p \int_{a+(i-1)\frac{b-a}{n}}^{a+i\frac{b-a}{n}} (x - x_i)^{p+1} dx\right|$$
$$= \frac{M}{(p+2)!} \sum_{i=1}^{n} n^p (x - x_i)^{p+2} \Big|_{a+(i-1)\frac{b-a}{n}}^{a+i\frac{b-a}{n}}$$
$$= \frac{M(b-a)^{p+2}}{(p+2)!n} \to 0 \quad (n \to \infty)$$

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} n^p \left\{ \left[\sum_{k=0}^{p-1} (-1)^k \cdot \frac{(b-a)^{k+1}}{k! \cdot (k+1) n^{k+1}} \sum_{i=1}^{n} f^{(k)}(x_i) \right] - \int_a^b f(x)\, dx \right\} \\
&= -\lim_{n\to\infty} \sum_{i=1}^{n} n^p \int_{a+(i-1)\frac{b-a}{n}}^{a+i\frac{b-a}{n}} \frac{f^{(p)}(x_i)}{p!} (x-x_i)^p\, dx \\
&= -\lim_{n\to\infty} \frac{1}{(p+1)!} \sum_{i=1}^{n} n^p f^{(p)}(x_i)(x-x_i)^{p+1} \Big|_{a+(i-1)\frac{b-a}{n}}^{a+i\frac{b-a}{n}} \\
&= \lim_{n\to\infty} \frac{(-1)^{p+1}(b-a)^{p+1}}{(p+1)! \cdot n} \sum_{i=1}^{n} f^{(p)}\left(a + i\frac{b-a}{n}\right) \\
&= \frac{(-1)^{p+1}(b-a)^p}{(p+1)!} \int_a^b f^{(p)}(x)\, dx \\
&= \frac{(-1)^{p+1}(b-a)^p}{(p+1)!} \left[f^{(p-1)}(b) - f^{(p-1)}(a) \right]
\end{aligned}$$

Q.E.D. Although the discussion is about adding borders to sequences, the same concept can also be applied to function limits.

[e.g. 27.2] Compute:

$$\lim_{x\to\infty} x \cdot \left(\left(\frac{a^{\frac{1}{x}} + b^{\frac{1}{x}}}{2} \right)^x - \sqrt{ab} \right) \quad (a, b > 0)$$

Solution: We already know from Equivalent Infinitesimals **[e.g. 6.2]** that

$$\lim_{x\to\infty} \left(\frac{a^{\frac{1}{x}} + b^{\frac{1}{x}}}{2} \right)^x = \sqrt{ab} \quad (a, b > 0)$$

Therefore,

$$\begin{aligned}
&\lim_{x\to\infty} x \cdot \left(\left(\frac{a^{\frac{1}{x}} + b^{\frac{1}{x}}}{2} \right)^x - \sqrt{ab} \right) \\
&= \lim_{t\to 0} \frac{\left(\frac{a^t+b^t}{2} \right)^{\frac{1}{t}} - \sqrt{ab}}{t} \\
&= \sqrt{ab} \lim_{t\to 0} \frac{e^{\frac{1}{t}\ln\left(\frac{a^t+b^t}{2}\right) - \frac{1}{2}\ln ab} - 1}{t}
\end{aligned}$$

Using equivalent infinitesimals, we have

$$\begin{aligned}
&\sqrt{ab} \lim_{t\to 0} \frac{e^{\frac{1}{t}\ln\left(\frac{a^t+b^t}{2}\right) - \frac{1}{2}\ln ab} - 1}{t} \\
&= \sqrt{ab} \lim_{t\to 0} \frac{\frac{1}{t}\ln\left(\frac{a^t+b^t}{2}\right) - \frac{1}{2}\ln ab}{t} \\
&= \sqrt{ab} \lim_{t\to 0} \frac{\ln\left(\frac{a^t+b^t}{2}\right) - \frac{t}{2}\ln ab}{t^2} \\
&= \sqrt{ab} \lim_{t\to 0} \frac{\ln\left(\frac{1}{2} \cdot \frac{a^t+b^t}{a^{\frac{t}{2}} \cdot b^{\frac{t}{2}}}\right)}{t^2} \\
&= \sqrt{ab} \lim_{t\to 0} \frac{\ln\left(\frac{1}{2} \cdot \frac{(a^{\frac{t}{2}} - b^{\frac{t}{2}})^2}{a^{\frac{t}{2}} \cdot b^{\frac{t}{2}}} + 1\right)}{t^2}
\end{aligned}$$

Using equivalent infinitesimals again, we obtain:

$$
\begin{aligned}
&= \sqrt{ab}\lim_{t\to 0}\frac{\ln\left(\frac{1}{2}\cdot\frac{(a^{\frac{t}{2}}-b^{\frac{t}{2}})^2}{a^{\frac{t}{2}}\cdot b^{\frac{t}{2}}}+1\right)}{t^2}\\
&= \sqrt{ab}\lim_{t\to 0}\frac{\frac{1}{2}\cdot\frac{(a^{\frac{t}{2}}-b^{\frac{t}{2}})^2}{a^{\frac{t}{2}}\cdot b^{\frac{t}{2}}}}{t^2}\\
&= \frac{\sqrt{ab}}{2}\lim_{t\to 0}\left(\frac{a^{\frac{t}{2}}-b^{\frac{t}{2}}}{t}\right)^2\\
&= \frac{\sqrt{ab}}{2}\left(\lim_{t\to 0}\frac{a^{\frac{t}{2}}-1-(b^{\frac{t}{2}}-1)}{t}\right)^2\\
&= \frac{\sqrt{ab}}{2}\left(\frac{\ln b-\ln a}{2}\right)^2\\
&= \boxed{\frac{\sqrt{ab}}{8}\ln^2\left(\frac{a}{b}\right)}
\end{aligned}
$$

Since the problem of adding borders is relatively simple, we can even create our own problems.

[e.g. 27.3] Compute:

$$\lim_{n\to\infty} n\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma\right)$$

Solution: We know that $\sum_{k=1}^{n}\frac{1}{k}=\ln n+\gamma+o(1)$. Therefore,

$$
\begin{aligned}
&\lim_{n\to\infty} n\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma\right)\\
&= \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma}{\frac{1}{n}}\\
&= \lim_{n\to\infty}\frac{\left(\sum_{k=1}^{n+1}\frac{1}{k}-\ln(n+1)-\gamma\right)-\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma\right)}{\frac{1}{n+1}-\frac{1}{n}}\\
&= -\lim_{n\to\infty} n(n+1)\left(\frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right)\right)
\end{aligned}
$$

Since $\ln\left(1+\frac{1}{n}\right)=\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)$, then

$$
\begin{aligned}
&-\lim_{n\to\infty} n(n+1)\left(\frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right)\right)\\
&= -\lim_{n\to\infty} n(n+1)\left(\frac{1}{n+1}-\frac{1}{n}+\frac{1}{2}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right)\\
&= -\lim_{n\to\infty} n(n+1)\left(\frac{1}{n+1}-\frac{1}{n}\right)-\lim_{n\to\infty} n(n+1)\left(\frac{1}{2}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right)\\
&= \boxed{\frac{1}{2}}
\end{aligned}
$$

We can continue to add borders and compute further.

[e.g.27.4] Calculation:

$$\lim_{n\to\infty} n\left\{n\left[\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma-\frac{1}{2}\right]\right\}$$

Solution:

$$\begin{aligned}
&\lim_{n\to\infty} n\left\{n\left[\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma-\frac{1}{2}\right]\right\}\\
&=\lim_{n\to\infty}\frac{\left[\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma-\frac{1}{2n}\right]}{\frac{1}{n^2}}\\
&=\lim_{n\to\infty}\frac{\left[\sum_{k=1}^{n+1}\frac{1}{k}-\ln(n+1)-\gamma-\frac{1}{2(n+1)}\right]-\left[\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma-\frac{1}{2n}\right]+\frac{1}{2n}-\frac{1}{2(n+1)}}{\frac{1}{(n+1)^2}-\frac{1}{n^2}}\\
&=-\lim_{n\to\infty} n^2(n+1)^2\frac{\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)-\ln\left(1+\frac{1}{n}\right)}{2n+1}\\
&=-\lim_{n\to\infty}\frac{n^3}{2}\left[\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)-\ln\left(1+\frac{1}{n}\right)\right]
\end{aligned}$$

Since

$$\ln\left(1+\frac{1}{n}\right)=\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n^2}+\frac{1}{3}\cdot\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)$$

then

$$\begin{aligned}
&-\lim_{n\to\infty}\frac{n^3}{2}\left[\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)-\ln\left(1+\frac{1}{n}\right)\right]\\
&=-\lim_{n\to\infty}\frac{n^3}{2}\left[\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)-\frac{1}{n}+\frac{1}{2}\cdot\frac{1}{n^2}-\frac{1}{3}\cdot\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)\right]\\
&=-\lim_{n\to\infty}\frac{n^3}{2}\left[-\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{2}\cdot\frac{1}{n^2}\right]+\frac{1}{6}\\
&=\lim_{n\to\infty}\frac{n^3}{4}\left[\frac{1}{n(n+1)}-\frac{1}{n^2}\right]+\frac{1}{6}\\
&=\lim_{n\to\infty}\frac{n^3}{4}\left[-\frac{1}{n^2(n+1)}\right]+\frac{1}{6}\\
&=-\frac{1}{4}+\frac{1}{6}\\
&=\boxed{-\frac{1}{12}}
\end{aligned}$$

Meanwhile, we also obtain a higher-precision estimation:

$$\sum_{k=1}^{n}\frac{1}{k}=\ln n+\gamma+\frac{1}{2n}-\frac{1}{12n^2}+o\left(\frac{1}{n^2}\right)$$

In fact, according to the Euler-Maclaurin formula:

$$\sum_{k=a}^{b}f(k)=\int_a^b f(x)dx+\frac{f(a)+f(b)}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}\left[f^{(2n-1)}(b)-f^{(2n-1)}(a)\right]$$

Let $a=1, b=n, f(x)=\frac{1}{x}$, then:

$$\sum_{k=1}^{n}\frac{1}{k}=\ln n+\frac{1+\frac{1}{n}}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}\left[f^{(2n-1)}(b)-f^{(2n-1)}(a)\right]=\ln n+\gamma+\frac{1}{2n}-\sum_{k=1}^{\infty}\frac{B_{2k}}{2k\cdot n^{2k}}$$

For the detailed proof, see: https://zhuanlan.zhihu.com/p/148221397
Therefore, readers can prove by themselves:

$$\lim_{n\to\infty} n\left\{n\left[n\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma\right)-\frac{1}{2}\right]+\frac{1}{12}\right\}=0$$

[**e.g.27.5**] Calculation:

$$\lim_{n\to\infty} n\sin(2\pi e n!)$$

Solution: Consider

$$e=\sum_{k=1}^{\infty}\frac{1}{k!}=\sum_{k=1}^{n}\frac{1}{k!}+\sum_{j=n+1}^{\infty}\frac{1}{j!}$$

Then

$$\begin{aligned}
&\lim_{n\to\infty} n\sin(2\pi e n!)\\
&=\lim_{n\to\infty} n\sin\left[2\pi\left(\sum_{k=1}^{n}\frac{1}{k!}+\sum_{j=n+1}^{\infty}\frac{1}{j!}\right)n!\right]\\
&=\lim_{n\to\infty} n\sin\left[2\pi\left(\sum_{j=n+1}^{\infty}\frac{1}{j!}\right)n!+2\pi\left(1+n+\ldots+n!\right)\right]\\
&=\lim_{n\to\infty} n\sin\left[2\pi\left(\frac{1}{(n+1)!}+o\left(\frac{1}{(n+1)!}\right)\right)n!\right]\\
&=\lim_{n\to\infty} n\sin\left(\frac{2\pi}{n+1}+o\left(\frac{1}{n+1}\right)\right)\\
&=\lim_{n\to\infty} n\left(\frac{2\pi}{n+1}+o\left(\frac{1}{n+1}\right)\right)\\
&=\boxed{2\pi}
\end{aligned}$$

[**e.g.27.6**] Calculation:

$$\lim_{n\to\infty} n^2\left[n\sin(2\pi e n!)-2\pi\right]$$

Solution: Consider

$$e=\sum_{k=1}^{\infty}\frac{1}{k!}=\sum_{k=1}^{n}\frac{1}{k!}+\sum_{j=n+1}^{\infty}\frac{1}{j!}$$

Then

$$\begin{aligned}
&\lim_{n\to\infty} n^2\left[n\sin\left(2\pi e n!\right)-2\pi\right]\\
&=\lim_{n\to\infty} n^2\left[n\sin\left(2\pi\left(\sum_{k=1}^{n}\frac{1}{k!}+\sum_{j=n+1}^{\infty}\frac{1}{j!}\right)n!\right)-2\pi\right]\\
&=\lim_{n\to\infty} n^2\left[n\sin\left(2\pi\left(\sum_{j=n+1}^{\infty}\frac{1}{j!}\right)n!+2\pi(1+n+\ldots+n!)\right)-2\pi\right]\\
&=\lim_{n\to\infty} n^2\left[n\sin\left(\frac{2\pi}{n+1}+\frac{2\pi}{(n+1)(n+2)}+\frac{2\pi}{(n+1)(n+2)(n+3)}+o\left(\frac{1}{n^3}\right)\right)-2\pi\right]
\end{aligned}$$

Since

$$\sin x = x-\frac{1}{6}x^3+o(x^3)$$

therefore

$$\begin{aligned}
&\sin\left(\frac{2\pi}{n+1}+\frac{2\pi}{(n+1)(n+2)}+\frac{2\pi}{(n+1)(n+2)(n+3)}+o\left(\frac{1}{n^3}\right)\right)\\
&=\frac{2\pi}{n+1}+\frac{2\pi}{(n+1)(n+2)}+\frac{2\pi}{(n+1)(n+2)(n+3)}-\frac{1}{6}\cdot\frac{8\pi^3}{(n+1)^3}+o\left(\frac{1}{n^3}\right)
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{n\to\infty} n^2\left[n\sin\left(\frac{2\pi}{n+1}+\frac{2\pi}{(n+1)(n+2)}+\frac{2\pi}{(n+1)(n+2)(n+3)}+o\left(\frac{1}{n^3}\right)\right)-2\pi\right]\\
&=\lim_{n\to\infty} n^2\left[\frac{2\pi n}{n+1}+\frac{2\pi n}{(n+1)(n+2)}+\frac{2\pi n}{(n+1)(n+2)(n+3)}-\frac{n}{6}\cdot\frac{8\pi^3}{(n+1)^3}+o\left(\frac{1}{n^2}\right)-2\pi\right]\\
&=2\pi-\frac{4}{3}\pi^3+2\pi\lim_{n\to\infty} n^2\left[\frac{n}{n+1}+\frac{n}{(n+1)(n+2)}-1\right]\\
&=2\pi-\frac{4}{3}\pi^3-2\pi\lim_{n\to\infty}\frac{2n^2}{(n+1)(n+2)}\\
&=\boxed{-2\pi-\frac{4}{3}\pi^3}
\end{aligned}$$

28 Iterated Limits

An iterated limit is a type of limit for multivariate functions where the limit is taken sequentially with respect to each variable. The general approach to solving it is to fix one variable, find the inner limit first, and then the outer limit. Functions of three or more variables are similar to bivariate functions, so the following examples focus on bivariate functions.

Iterated limits generally cannot be exchanged in order, but they can be exchanged under certain specific conditions. The following theorem provides a condition that guarantees the exchangeability of iterated limits.

Theorem 1 Suppose $f(x,y)$ is defined on some neighborhood $U^0(P_0)$ of $P_0(x_0,y_0)$. If:
(1) For any $y \neq y_0$ in $U^0(P_0)$, $\lim_{x\to x_0} f(x,y) = g(y)$;
(2) $\lim_{x\to x_0} f(x,y)$ converges uniformly in x over $U^0(P_0)$: $\lim_{x\to x_0} f(x,y) = h(x)$.
Then

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y) = \lim_{y\to y_0}\lim_{x\to x_0} f(x,y).$$

[e.g.28.1] Calculate:

$$\lim_{x\to\infty}\lim_{r\to 0}\frac{\left[(x+1)^{r+1}-x^{r+1}\right]^{\frac{1}{r}}}{x}$$

Solution:

$$\begin{aligned}
&\lim_{x\to\infty}\lim_{r\to 0}\frac{\left[(x+1)^{r+1}-x^{r+1}\right]^{\frac{1}{r}}}{x}\\
&=\lim_{x\to\infty}\lim_{r\to 0}\frac{\left[(r+1)\xi^{r}\right]^{\frac{1}{r}}}{x}\quad (x\le\xi\le x+1)\\
&=\lim_{x\to\infty}\frac{\xi}{x}\lim_{r\to 0}(r+1)^{\frac{1}{r}}\\
&=\boxed{e}
\end{aligned}$$

[e.g.28.2] Calculate:

$$\lim_{t\to 0^+}\lim_{x\to+\infty}\frac{\displaystyle\int_0^{\sqrt{t}} dx\int_{x^2}^{t}\sin y^3 dy}{\left[\left(\frac{2}{\pi}\cdot\arctan\frac{x}{t^2}\right)^x-1\right]\arctan t^{\frac{5}{2}}}$$

Solution: First, observe the numerator and denominator. Notice that the double integral in the numerator does not actually contain x; x is just an intermediate variable that will eventually cancel out. Therefore, when taking the limit with respect to x, only the term in the denominator containing x needs to be considered.

$$\begin{aligned}
&\lim_{x\to+\infty}\left[\left(\frac{2}{\pi}\cdot\arctan\frac{x}{t^2}\right)^x-1\right]\\
&=\lim_{x\to+\infty}\left(\frac{2}{\pi}\cdot\arctan\frac{x}{t^2}\right)^x-1\\
&=e^{\lim_{x\to+\infty} x\ln\left(\frac{2}{\pi}\cdot\arctan\frac{x}{t^2}\right)}-1
\end{aligned}$$

Next, calculate the limit of the exponent:

$$\begin{aligned}
&\lim_{x\to+\infty} x\ln\left(\frac{2}{\pi}\cdot\arctan\frac{x}{t^2}\right)\\
&=\lim_{s\to0^+}\frac{\ln\left(\frac{2}{\pi}\cdot\arctan\frac{1}{st^2}-1+1\right)}{s}\\
&=\lim_{s\to0^+}\frac{\frac{2}{\pi}\cdot\arctan\frac{1}{st^2}-1}{s}\\
&=\frac{2}{\pi}\cdot\lim_{s\to0^+}\frac{\arctan\frac{1}{st^2}-\frac{\pi}{2}}{s}\\
&=\frac{2}{\pi}\cdot\lim_{s\to0^+}\frac{-\frac{1}{s^2t^2+\frac{1}{t^2}}}{1}\\
&=-\frac{2}{\pi}t^2
\end{aligned}$$

Now consider the double integral in the numerator. The integration region is $x^2\le y\le t$, $0\le x\le\sqrt{t}$. By

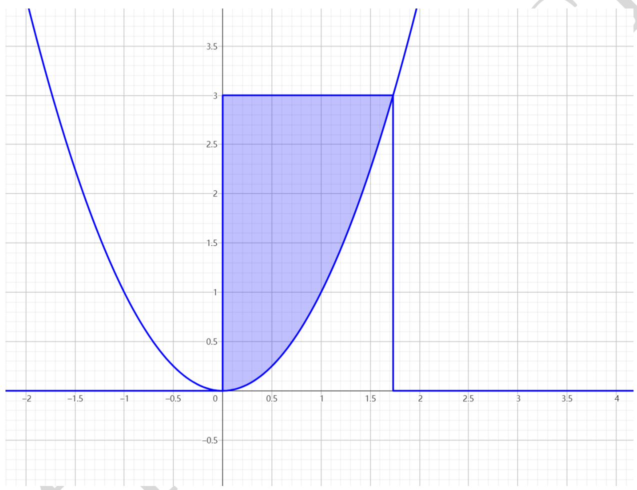

Figure 2: Integration Region

changing the order of integration, we get

$$\int_0^{\sqrt{t}}dx\int_{x^2}^{t}\sin y^3dy=\int_0^t dy\int_0^{\sqrt{y}}\sin y^3dx=\int_0^t\sqrt{y}\sin y^3dy$$

Therefore,

$$
\begin{aligned}
&\lim_{t\to 0^+}\lim_{x\to+\infty}\frac{\displaystyle\int_0^{\sqrt{t}}dx\int_{x^2}^{t}\sin y^3dy}{\left[\left(\dfrac{2}{\pi}\cdot\arctan\dfrac{x}{t^2}\right)^x-1\right]\arctan t^{\frac{5}{2}}}\\
=&\lim_{t\to 0^+}\frac{\int_0^t\sqrt{y}\sin y^3dy}{\left[e^{-\frac{2}{\pi}t^2}-1\right]\arctan t^{\frac{5}{2}}}\\
=&\lim_{t\to 0^+}\frac{\int_0^t\sqrt{y}\sin y^3dy}{-\frac{2}{\pi}t^2\cdot t^{\frac{5}{2}}}\\
=&\lim_{t\to 0^+}\frac{\sqrt{t}\sin t^3}{-\frac{2}{\pi}\cdot\frac{9}{2}t^{\frac{7}{2}}}\\
=&\boxed{-\frac{\pi}{9}}
\end{aligned}
$$

[e.g.28.3 (Difficult)] Calculate:

$$
\lim_{x\to 0^+}\lim_{y\to+\infty}\frac{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{m=0}^{\infty}\frac{1}{n\cdot 2^m+1}\int_0^{x^2}\frac{\pi(\sqrt[4]{1+t}-1)\sin t^4}{\sum_{n=1}^{\infty}\frac{((n-1)!)^2(2t)^{2n}}{(2n)!}\int_0^1\frac{(1-2x)\ln(1-x)}{x^2-x+1}\mathrm{d}x}\mathrm{d}t}{x^2(x-\tan x)\ln(x^2+1)\left[\left(\frac{2\arctan\frac{y}{x}}{\pi}\right)^y-1\right]}
$$

Note: This problem looks very intimidating and complex. In fact, it can be solved by breaking it down step by step. The biggest difficulty of this problem is to calculate $\sum_{n=1}^{\infty}\frac{((n-1)!)^2(2t)^{2n}}{(2n)!}$.

Solution:

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{m=0}^{\infty}\frac{1}{n\cdot 2^m+1}\\
=&\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n-1}}{n}\frac{1}{n\cdot 2^m+1}\\
=&\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 x^{n\cdot 2^m}dx\\
=&\int_0^1\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x^{2^m})^n\right)dx\\
=&\int_0^1\left(\sum_{m=0}^{\infty}\ln(x^{2^m}+1)\right)dx
\end{aligned}
$$

Notice that

$$
\sum_{m=0}^{\infty}\ln(x^{2^m}+1)=\ln\left(\prod_{m=0}^{\infty}(x^{2^m}+1)\right)=\ln\left(\sum_{k=0}^{\infty}x^k\right)=-\ln(1-x)\quad(|x|<1)
$$

Therefore,

$$
\int_0^1\left(\sum_{m=0}^{\infty}\ln(x^{2^m}+1)\right)dx=-\int_0^1\ln(1-x)dx=1
$$

And

$$\begin{aligned}
&\int_0^1 \frac{(1-2x)\ln(1-x)}{x^2-x+1}\mathrm{d}x \\
&= -\int_0^1 \ln(1-x)d\left(\ln(x^2-x+1)\right) \\
&= -\ln(1-x)\ln(x^2-x+1)\big|_0^1 + \int_0^1 \frac{\ln(x^2-x+1)}{x-1}dx \\
&= \int_0^1 \frac{\ln(t^2-t+1)}{-t}dt \quad (t=1-x) \\
&= \int_0^1 \frac{\ln(t^3+1)-\ln(1+t)}{-t}dt \\
&= -\frac{1}{3}\int_0^1 \frac{\ln(1+s)}{s}ds \quad (s=x^3) + \int_0^1 \frac{\ln(1+x)}{x}dx \\
&= \frac{2}{3}\int_0^1 \frac{\ln(1+x)}{x}dx \\
&= \frac{2}{3}\int_0^1 \sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^{k-1}}{k}dx \\
&= \frac{2}{3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} \\
&= \frac{2}{3}\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - 2\sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right) \\
&= \frac{2}{3}\cdot\frac{1}{2}\cdot\frac{\pi^2}{6} \\
&= \frac{\pi^2}{18}
\end{aligned}$$

Observing the numerator and denominator, only the denominator contains y, so we only need to take the limit of the term containing y in the denominator.

$$\begin{aligned}
&\lim_{y\to+\infty}\left(\left(\frac{2\arctan\frac{y}{x}}{\pi}\right)^y - 1\right) \\
&= \lim_{y\to+\infty}\left(\frac{2\arctan\frac{y}{x}}{\pi}\right)^y - 1 \\
&= e^{\lim_{y\to+\infty} y\ln\left(\frac{2\arctan\frac{y}{x}}{\pi}\right)} - 1
\end{aligned}$$

Next, we calculate the limit of the exponent:

$$\begin{aligned}
&\lim_{y\to+\infty} y\ln\left(\frac{2\arctan\frac{y}{x}}{\pi}\right)\\
&=\lim_{s\to0^+}\frac{\ln\left(\frac{2\arctan\frac{1}{sx}}{\pi}\right)}{s}\\
&=\lim_{s\to0^+}\frac{\ln\left(\frac{2\arctan\frac{1}{sx}}{\pi}-1+1\right)}{s}\\
&=\lim_{s\to0^+}\frac{\frac{2\arctan\frac{1}{sx}}{\pi}-1}{s}\\
&=\frac{1}{\pi}\lim_{s\to0^+}\frac{-2\frac{1}{s^2+\frac{1}{x}}}{1}\\
&=-\frac{2}{\pi}x
\end{aligned}$$

Therefore,

$$\lim_{y\to+\infty}\left(\left(\frac{2\arctan\frac{y}{x}}{\pi}\right)^y-1\right)=e^{-\frac{2}{\pi}x}-1$$

Finally, consider the series

$$\sum_{n=1}^{\infty}\frac{((n-1)!)^2(2t)^{2n}}{(2n)!}$$

The convergence domain of the series $\sum_{n=1}^{\infty}\frac{((n-1)!)^2(2t)^{2n}}{(2n)!}$ is $[-1,1]$. Let $S(x)=\sum_{n=1}^{\infty}\frac{((n-1)!)^2(2x)^{2n}}{(2n)!}$, then

$$S'(x)=\sum_{n=1}^{\infty}\frac{((n-1)!)^2}{(2n)!}(4n)(2x)^{2n-1},\ -1<x<1,$$

$$S''(x)=\sum_{n=1}^{\infty}\frac{((n-1)!)^2}{(2n)!}(8n)(2n-1)(2x)^{2n-2},\ -1<x<1,$$

Thus,

$$\begin{aligned}
&-xS'(x)+\left(1-x^2\right)S''(x)\\
&=-\sum_{n=1}^{\infty}\frac{((n-1)!)^2}{(2n)!}(2n)(2x)^{2n}+\sum_{n=1}^{\infty}\frac{((n-1)!)^2}{(2n)!}(8n)(2n-1)(2x)^{2n-2}-\sum_{n=1}^{\infty}\frac{((n-1)!)^2}{(2n)!}(2n)(2n-1)(2x)^{2n}\\
&=4+\sum_{n=2}^{\infty}\frac{((n-1)!)^2}{(2n)!}(8n)(2n-1)(2x)^{2n-2}-\sum_{n=1}^{\infty}\frac{((n-1)!)^2}{(2n)!}\left(4n^2\right)(2x)^{2n}\\
&=4+\sum_{n=1}^{\infty}\frac{((n)!)^2}{(2n+2)!}8(n+1)(2n+1)(2x)^{2n}-\sum_{n=1}^{\infty}\frac{((n-1)!)^2}{(2n)!}\left(4n^2\right)(2x)^{2n}\\
&=4
\end{aligned}$$

Therefore:

$$-xS'(x)+\left(1-x^2\right)S''(x)=4,\ -1<x<1,$$

Dividing both sides by $\sqrt{1-x^2}$, we get

$$-\frac{x}{\sqrt{1-x^2}}S'(x)+\sqrt{1-x^2}S''(x)=\frac{4}{\sqrt{1-x^2}},$$

Thus,

$$\sqrt{1-x^2}S'(x) = 4\arcsin x + C,$$

From $S'(0) = 0$, we get $C = 0$, so

$$S'(x) = \frac{4\arcsin x}{\sqrt{1-x^2}},$$

Integrating both sides, we get

$$S(x) = 2\text{arcsin}^2 x + C_1,$$

From $S(0) = 0$, we get $C_1 = 0$, then $S(x) = 2\text{arcsin}^2 x \, (-1 < x < 1)$, thus

$$\sum_{n=1}^{\infty} \frac{((n-1)!)^2 (2t)^{2n}}{(2n)!} = 2\text{arcsin}^2 t, \ -1 < t < 1.$$

Then

$$\lim_{x\to 0^+} \lim_{y\to+\infty} \frac{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{m=0}^{\infty} \frac{1}{n\cdot 2^m+1} \int_0^{x^2} \frac{\pi(\sqrt[4]{1+t}-1)\sin t^4}{\sum_{n=1}^{\infty} \frac{((n-1)!)^2(2t)^{2n}}{(2n)!} \int_0^1 \frac{(1-2x)\ln(1-x)}{x^2-x+1}\mathrm{d}x} \,\mathrm{d}t}{x^2(x-\tan x)\ln(x^2+1)\left[\left(\frac{2\arctan\frac{y}{x}}{\pi}\right)^y - 1\right]}$$

$$= \lim_{x\to 0^+} \frac{\int_0^{x^2} \frac{\pi(\sqrt[4]{1+t}-1)\sin t^4}{2\text{arcsin}^2 t\cdot\frac{\pi^2}{18}} dt}{x^2(-\frac{1}{3}x^3)x^2\left(-\frac{2}{\pi}x\right)}$$

$$= \frac{27}{16} \lim_{x\to 0^+} \frac{2x\frac{\sqrt[4]{1+x^2}-1}{\arcsin^2(x^2)}\sin x^8}{x^7}$$

$$= \frac{27}{8} \lim_{x\to 0^+} \frac{\sqrt[4]{1+x^2}-1}{x^2}$$

$$= \frac{27}{8} \cdot \frac{1}{4}$$

$$= \boxed{\frac{27}{32}}$$

29 Double Limit

Double limits and iterated limits are related but distinct. A double limit considers the limit as the independent variables simultaneously tend to a certain point, while an iterated limit is the limit of a variable tending to a value sequentially. There are some connections between them:

Theorem 1
Suppose the double limit $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = A$. When $y \neq b$, if $\lim_{x\to a} f(x,y)$ exists, then

$$\lim_{y\to b}\lim_{x\to a} f(x,y) = A$$

29.1 Definition Method

Definition 1 Suppose the n-variable function $f(x_1, x_2, \cdot, x_n)$ is defined in a punctured neighborhood of a, and A is a constant. If for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $0 < |x - a| < \delta$, we have

$$|f(x)| < \varepsilon$$

then the limit of the n-variable function $f(x)$ is A, denoted as $\lim_{x\to a} f(x) = A$.

[e.g. 29.1.1] Calculate:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$$

Solution: For all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $0 < \sqrt{x^2+y^2} < \delta$, we have

$$\left|\frac{xy}{x+y}\right| \leq \left|\frac{\sqrt{xy}}{2}\right| \leq \sqrt{\frac{x^2+y^2}{2}} \leq \delta$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y} = 0$$

29.2 Polar Coordinates

Polar coordinates are also a good method to calculate double limits, but most of the time, polar coordinates are more suitable for proving the non-existence of double limits, and may lead to errors when calculating double limits. Although polar coordinate substitution is feasible for calculating double limits, because we have a theorem:

Theorem 2 If the domain of the binary function is D, and $P_0(x_0, y_0)$ is a limit point of D. Then the necessary and sufficient condition for the double limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = A$ is: Let $x = x_0 + r\cos\theta, y = y_0 + r\sin\theta$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any $0 < r < \delta, 0 \leq \theta \leq 2\pi$, $(r,\theta) \in D$, we have

$$|f(x_0 + r\cos\theta, y_0 + r\sin\theta) - A| < \varepsilon$$

For the proof, see: https://zhuanlan.zhihu.com/p/503998713

However, some people ignore the arbitrariness of the path taken in double integrals when calculating limits after polar coordinate substitution, leading to errors. Here is an example to illustrate this:

[e.g. 29.2.1] Calculate:

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y}$$

Solution: Substitute $x = r\cos\theta, y = r\sin\theta$

Therefore

$$\begin{aligned} &\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y} \\ =&\lim_{r\to 0} \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2\cos^2\theta + r\sin\theta} \\ =&\lim_{r\to 0} \frac{r^2(\cos^3\theta + \sin^3\theta)}{r\cos^2\theta + \sin\theta} \end{aligned}$$

If $\sin\theta = 0$, then

$$\lim_{r\to 0}\frac{r^2(\cos^3\theta+\sin^3\theta)}{r\cos^2\theta+\sin\theta} = \lim_{r\to 0} r\cos\theta = 0$$

If $\sin\theta \neq 0$, then

$$\lim_{r\to 0}\frac{r^2(\cos^3\theta+\sin^3\theta)}{r\cos^2\theta+\sin\theta} = \lim_{r\to 0}\frac{r^2(\cos^3\theta+\sin^3\theta)}{\sin\theta} = 0$$

Therefore,

$$\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y} = 0$$

But is this correct? If we take $y = -x^2 + x^3$, then

$$\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y} = \lim_{x\to 0}\frac{x^3+(-x^2+x^3)^3}{x^3} = 1$$

This contradicts the previous calculation! Why?

Since Theorem 2 is correct, there must be an error in the calculation process.

In fact, during the previous calculation, we did not ensure the arbitrariness of θ. In other words, when θ tends to 0, the limit $\lim_{r\to 0}\frac{r^2(\cos^3\theta+\sin^3\theta)}{\sin\theta}$ is indeterminate. We cannot directly conclude that the result is 0, so there is an error in the final step of calculating the limit. Only the limit of an infinitesimal multiplied by a bounded quantity is 0.

[**e.g. 29.2.2**] Calculate:

$$\lim_{(x,y)\to(0,0)}\frac{xy\sin(ky)}{x^2+y^4}$$

Solution: Let $x = r\cos\theta, y^2 = r\sin\theta, y = \pm\sqrt{r\sin\theta}, \theta\in(0,\pi)$

Then

$$\begin{aligned}&\lim_{(x,y)\to(0,0)}\frac{xy\sin(ky)}{x^2+y^4}\\ &= \lim_{r\to 0^+}\frac{\pm r^2\sin\theta\cos\theta\sin k\sqrt{r\sin\theta}}{r^2}\\ &= \lim_{r\to 0^+}\pm\sin\sqrt{r(k\sqrt{\sin\theta})}\sin\theta\cos\theta\\ &= \boxed{0}\end{aligned}$$

29.3 Squeeze Theorem

[**e.g.29.3.1**] Calculate:

$$\lim_{(x,y)\to(0,0)}\frac{\sin(x^2y+y^4)}{x^2+y^2}$$

Solution:

$$0 \le \left|\frac{\sin(x^2y+y^4)}{x^2+y^2}\right| \le \left|\frac{x^2y+y^4}{x^2+y^2}\right| \le \frac{x^2}{x^2+y^2}|y| + \frac{y^4}{x^2+y^2} \le |y| + y^2 \to 0 \text{ as } (x,y)\to(0,0)$$

[e.g.29.3.2] Calculate:

$$\lim_{(x,y)\to(+\infty,+\infty)}\left(\frac{xy}{x^2+y^2}\right)^{x^2}$$

Solution: Since

$$0\le\frac{xy}{x^2+y^2}\le\frac{\frac{x^2+y^2}{2}}{x^2+y^2}\cdot\frac{1}{2}$$

therefore

$$0\le\left(\frac{xy}{x^2+y^2}\right)^{x^2}\le\left(\frac{1}{2}\right)^{x^2}\to 0\text{ as }(x,y)\to(+\infty,+\infty)$$

Hence

$$\lim_{(x,y)\to(+\infty,+\infty)}\left(\frac{xy}{x^2+y^2}\right)^{x^2}=\boxed{0}$$

[e.g.29.3.3] Calculate:

$$\lim_{(x,y)\to(0,0)}\frac{xy\sin(ky)}{x^2+y^4}$$

Solution:

$$0\le\left|\frac{xy\sin(ky)}{x^2+y^4}\right|\le\left|\frac{xy^2}{2xy^2}\right|\cdot|\sin ky|=\frac{1}{2}|\sin ky|\to 0\text{ as }(x,y)\to(0,0)$$

Thus

$$\lim_{(x,y)\to(0,0)}\frac{xy\sin(ky)}{x^2+y^4}=\boxed{0}$$

29.4 Holistic Approach

The holistic approach involves treating certain elements as a whole, such as $xy, x+y, x-y, \frac{x}{y}$, etc. If they satisfy certain properties (e.g., tending to 0), then the whole can be used with important limits, equivalent infinitesimals, Taylor's formula, etc., similar to single-variable limits.

29.4.1 Important Limits

[e.g.29.4.1.1] Calculate:

$$\lim_{(x,y)\to(0,0)}\frac{\sin x\sin y}{xy}$$

Solution:

$$\lim_{(x,y)\to(0,0)}\frac{\sin x\sin y}{xy}=\lim_{(x,y)\to(0,0)}\frac{x\cdot y}{xy}=\boxed{1}$$

[e.g.29.4.1.2] Calculate:

$$\lim_{(x,y)\to(0,0)}\frac{\sin(xy^2)\sin(x^2y)}{x^2y^2\sin(xy)}$$

Solution:

$$\lim_{(x,y)\to(0,0)}\frac{\sin(\frac{xy^2}{2})\sin(\frac{x^2y}{3})}{x^2y^2\sin(xy)}=\frac{1}{6}\lim_{(x,y)\to(0,0)}\frac{(xy^2)(x^2y)}{x^2y^2(xy)}=\boxed{\frac{1}{6}}$$

[e.g.29.4.1.3] Calculate:

$$\lim_{(x,y)\to(+\infty,0)}\left(1+\frac{1}{x+y}\right)^{x}$$

Solution:

$$\lim_{(x,y)\to(+\infty,0)}\left(1+\frac{1}{x+y}\right)^{x}=\lim_{(x,y)\to(+\infty,0)}\left(1+\frac{1}{x+y}\right)^{x+y}=\boxed{e}$$

29.4.2 Equivalent Infinitesimals

[e.g.29.4.2.1] Calculate:

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{xy+1}-1}{xy}$$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{xy+1}-1}{xy} = \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{2}}{xy} = \boxed{\frac{1}{2}}$$

[e.g.29.4.2.2] Calculate:

$$\lim_{(x,y)\to(0,0)} \frac{\ln(1+xy)\cdot\sin(x+y)}{1-\cos(x+y)}$$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{\ln(1+xy)\cdot\sin(x+y)}{1-\cos(x+y)} = \lim_{(x,y)\to(0,0)} \frac{xy\cdot(x+y)}{\frac{1}{2}(x+y)^2} = \lim_{(x,y)\to(0,0)} \frac{xy}{\frac{1}{2}(x+y)}$$

Since

$$0 \le \left|\frac{xy}{\frac{1}{2}(x+y)}\right| \le \sqrt{xy} \to 0 \text{ as } (x,y)\to(0,0)$$

therefore

$$\lim_{(x,y)\to(0,0)} \frac{\ln(1+xy)\cdot\sin(x+y)}{1-\cos(x+y)} = \boxed{0}$$

29.4.3 Taylor's Formula

[e.g.29.4.3.1] Calculate:

$$\lim_{(x,y)\to(0,0)} \frac{(xy-\sin xy)(x+y)^2}{\frac{1}{x}+\frac{1}{y}-\ln(1+x+y)^{\frac{1}{xy}}\sin^4(xy)}$$

Solution:

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} \frac{(xy-\sin xy)(x+y)^2}{\frac{1}{x}+\frac{1}{y}-\ln(1+x+y)^{\frac{1}{xy}}\sin^4(xy)} \\
&= \lim_{(x,y)\to(0,0)} \frac{(xy-\sin xy)(x+y)^2}{\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}\ln(1+x+y)\right)x^4y^4} \\
&= \lim_{(x,y)\to(0,0)} \frac{(xy-\sin xy)(x+y)^2}{(x+y-\ln(1+x+y))x^3y^3} \\
&= \lim_{(x,y)\to(0,0)} \frac{(x+y)^2\left(\frac{(xy)^3}{6}+o(x^3y^3)\right)}{\left(\frac{(x+y)^2}{2}+o((x+y)^2)\right)x^3y^3} \\
&= \frac{\frac{1}{6}}{\frac{1}{2}} \\
&= \boxed{\frac{1}{3}}
\end{aligned}$$

30 Some Famous Conclusions

30.1 Famous Constants

30.1.1 π

Conclusion 1 (Leibniz Series)

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} = \frac{\pi}{4}$$

Proof: Consider

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad (|x| < 1)$$

Therefore,

$$\sum_{k=0}^{\infty} (-x^2)^k = \frac{1}{1+x^2} \quad (|x| < 1)$$

Integrating both sides from 0 to 1, we get

$$\int_0^1 \sum_{k=0}^{\infty} (-x^2)^k dx = \sum_{k=0}^{\infty} \int_0^1 (-x^2)^k dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}$$

And

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} = \frac{\pi}{4}$$

Conclusion 2 (Basel Problem)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Proof: Consider the infinite product expansion of $\sin x$

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(k\pi)^2}\right)$$

And the Taylor expansion of $\sin x$

$$\sin x = x - \frac{x^3}{6} + \cdots$$

Then

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(k\pi)^2}\right) = 1 - \frac{x^2}{6} + \cdots$$

Comparing the coefficients of the square terms, we immediately get

$$\sum_{k=1}^{\infty} -\frac{1}{k^2\pi^2} = -\frac{1}{6}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Conclusion 3 (Gaussian Integral)

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Proof: Since

$$\int_{-\infty}^{0} e^{-x^2} dx = \int_{0}^{+\infty} e^{-x^2} dx$$

Therefore, it suffices to prove

$$I = \int_{0}^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Method 1 (Double Integral):

$$\begin{aligned} I^2 &= \int_{0}^{+\infty} e^{-x^2} dx \int_{0}^{+\infty} e^{-x^2} dx \\ &= \int_{0}^{+\infty} e^{-x^2} dx \int_{0}^{+\infty} e^{-y^2} dy \\ &= \int_{0}^{+\infty} dx \int_{0}^{+\infty} e^{-(x^2+y^2)} dy \\ &= \int_{0}^{\frac{\pi}{2}} d\theta \int_{0}^{+\infty} \rho e^{-\rho^2} d\rho \\ &= \int_{0}^{\frac{\pi}{2}} -\frac{1}{2} e^{-\rho^2} \bigg|_{0}^{+\infty} d\theta \\ &= \frac{\pi}{4} \end{aligned}$$

Therefore, $I = \frac{\sqrt{\pi}}{2}$ i.e.

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Method 2 (Γ Function): Consider

$$\Gamma(s) = \int_{0}^{+\infty} t^{s-1} e^{-t} dt$$

Substitute $t = x^2, dt = 2xdx$ Therefore,

$$\Gamma(s) = \int_{0}^{+\infty} t^{s-1} e^{-t} dt = 2 \int_{0}^{+\infty} x^{2s-1} e^{-x^2} dx$$

Let $s = \frac{1}{2}$, then

$$I = \frac{\Gamma\left(\frac{1}{2}\right)}{2}$$

Consider the Euler's Reflection Formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

Let $x = \frac{1}{2}$, then

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Therefore, $I = \frac{\sqrt{\pi}}{2}$ i.e.

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Conclusion 4 (Dirichlet Integral)

$$\int_0^{+\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

We have already proven this in Riemann's Lemma [**e.g. 22.4**], so we will not elaborate further.

corollary 1 In fact, for any $p > 0$, we have

$$\int_0^{+\infty} \frac{\sin px}{x} dx = \frac{\pi}{2}$$

We only need to substitute $x = \frac{t}{p}, dx = \frac{1}{p} dt$.

[**e.g. 30.1.1.1**] Compute:

$$\int_0^{+\infty} \frac{\sin^2 x}{x^2} dx$$

Solution:

$$\begin{aligned}
&\int_0^{+\infty} \frac{\sin^2 x}{x^2} dx \\
&= \int_0^{+\infty} \sin^2 x d\left(-\frac{1}{x}\right) \\
&= -\left.\frac{\sin^2 x}{x}\right|_0^{+\infty} + \int_0^{+\infty} \frac{\sin 2x}{x} dx \\
&= \boxed{\frac{\pi}{2}}
\end{aligned}$$

30.1.2 e

Two definitions:

Definition 1 (Limit Definition)

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

Definition 2 (Series Definition)

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Theorem 1 (Euler's Formula)

$$e^{i\theta} = i \sin\theta + \cos\theta$$

30.1.3 ϕ

Definition 3 (Fibonacci Sequence) $F_1 = F_2 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

corollary 2

$$\phi = \frac{\sqrt{5}-1}{2} = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$$

30.1.4 γ

Definition 4 (Limit Definition)

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$$

Definition 5 (Improper Integral Definition)

$$\gamma = \int_1^{+\infty}\left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)dx$$

Proof:

$$\begin{aligned}
&\int_1^{+\infty}\left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)dx \\
&= \lim_{n\to\infty}\int_1^{n}\left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)dx \\
&= \lim_{n\to\infty}\left(\int_1^{n}\frac{1}{\lfloor x\rfloor}dx - \ln n\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}\int_k^{k+1}\frac{1}{k}dx - \ln n\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}\frac{1}{k} + \frac{1}{n} - \frac{1}{n} - \ln n\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \\
&= \gamma
\end{aligned}$$

corollary 3

$$-\gamma = \Gamma'(1) = \Psi(1)$$

30.1.5 G

Catalan's Constant, denoted by G, is defined as:

Definition 6 (Series Definition)

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

Definition 7 (Integral Definition)

$$G = -\int_0^{\frac{\pi}{4}} \ln(\tan x)dx = \int_0^1 \frac{\arctan x}{x}dx$$

[e.g.30.1.5.1] Discuss the convergence of $I(a)$ and calculate $I(1)$ when $k = 2$:

Where

$$I(a) = \int_0^{+\infty} \frac{\ln(1+ax)}{1+x^k}dx \text{ for } a \geq 0$$

Solution: First, $\frac{\ln(1+ax)}{1+x^k} \geq 0$

1. When $k \leq 1$, there exists $X = \max\{1, \frac{e-1}{a}\}$ such that for $x > X$,

$$\int_X^{+\infty} \frac{\ln(1+ax)}{1+x^k}dx \geq \int_X^{+\infty} \frac{1}{1+x}dx = +\infty$$

Therefore, the improper integral diverges.

2. When $k > 1$, then there exists $\varepsilon > 0$ such that $k > 1 + \varepsilon$. And there exists $M > 0$ such that $\ln(1+ax) \leq Mx^{\varepsilon}$. Consider the interval $[1, +\infty)$, then

$$\begin{aligned}
&\int_1^{+\infty} \frac{\ln(1+ax)}{1+x^k}dx \\
&< \int_1^{+\infty} \frac{Mx^{\varepsilon}}{x^k}dx \\
&= M \left. \frac{x^{\varepsilon-k+1}}{\varepsilon-k+1} \right|_1^{+\infty} \\
&= \frac{M}{\varepsilon-k+1}
\end{aligned}$$

Next, consider the interval $[0, 1]$. Since $k > 1$, thus $1 + x^k \geq 1$ and $\ln(1+ax) \leq M'$. Therefore,

$$\begin{aligned}
&\int_0^1 \frac{\ln(1+ax)}{1+x^k}dx \\
&< \int_0^1 \frac{M'}{1}dx \\
&= M'
\end{aligned}$$

Thus, $I(a)$ converges uniformly on $[0, +\infty)$ with respect to a. Therefore,

$$I'(a) = \int_0^{+\infty} \frac{x}{(1+x^k)(1+ax)}dx$$

When $k = 2$, we have

$$\begin{aligned}
I'(a) &= \int_0^{+\infty} \frac{x}{(1+x^2)(1+ax)} dx \\
&= \frac{1}{a} \int_0^{+\infty} \frac{(ax+1)-1}{(1+x^2)(1+ax)} dx \\
&= \frac{1}{a} \int_0^{+\infty} \frac{1}{1+x^2} dx - \frac{1}{a} \int_0^{+\infty} \frac{(1+x^2)-x^2}{(1+x^2)(ax+1)} dx \\
&= \lim_{t\to+\infty} \left[\frac{\pi}{2a} - \frac{1}{a} \int_0^t \frac{1}{ax+1} dx + \frac{1}{a^2} \int_0^t \frac{(ax^2+x)-x}{(1+x^2)(ax+1)} dx \right] \\
&= \lim_{t\to+\infty} \left[\frac{\pi}{2a} - \frac{1}{a} \int_0^t \frac{1}{ax+1} dx + \frac{1}{a^2} \int_0^t \frac{x}{1+x^2} dx - \frac{1}{a^2} \int_0^t \frac{x}{(1+x^2)(ax+1)} dx \right] \\
&= \frac{\pi}{2a} + \lim_{t\to+\infty} \left[\frac{\ln(1+at)}{a^2} - \frac{\ln(1+t^2)}{2a^2} \right] - \frac{1}{a^2} I'(a) \\
&= \frac{\pi}{2a} + \frac{\ln a}{a^2} - \frac{1}{a^2} I'(a)
\end{aligned}$$

Therefore,

$$I'(a) = \frac{\pi}{2} \cdot \frac{1}{a^2+1} + \frac{\ln a}{a^2+1}$$

Since $I(0) = 0$, then

$$I(a) = \int_0^a \left(\frac{\pi}{2} \cdot \frac{1}{x^2+1} + \frac{\ln x}{x^2+1} \right) dx = \frac{\pi}{2} \cdot \arctan a + \int_0^a \frac{\ln x}{x^2+1} dx$$

Thus,

$$\begin{aligned}
I(1) &= \frac{\pi^2}{8} + \int_0^1 \frac{\ln x}{x^2+1} dx \\
&= \arctan x \cdot \ln x|_0^1 - \int_0^1 \frac{\arctan x}{x} dx + \frac{\pi^2}{8} \\
&= \frac{\pi^2}{8} - G
\end{aligned}$$

30.1.6 A

Glaisher-Kinkelin Constant

Definition 8

$$A = \lim_{n\to\infty} \frac{1^1 2^2 \cdots n^n}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} \cdot e^{-\frac{n^2}{4}}}$$

[e.g.30.1.6.1] Solve the limit:

$$\lim_{n\to\infty}\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{1}{2}}}$$

Solution: Let $x_n=\lim_{n\to\infty}\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{1}{2}}}$

$$\begin{aligned}\ln x_n&=\frac{1}{n}\sum_{k=1}^{n}\ln\binom{n}{k}-\frac{n}{2}+\frac{\ln n}{2}\\&=\frac{1}{n}\left(n\ln n!-\sum_{k=1}^{n}\ln k!-\sum_{k=1}^{n}\ln(n-k)!\right)-\frac{n}{2}+\frac{\ln n}{2}\\&=\frac{1}{n}\left((n+1)\ln n!-2\sum_{k=1}^{n}\ln k!\right)-\frac{n}{2}+\frac{\ln n}{2}\end{aligned}$$

Since

$$\sum_{k=1}^{n}\ln k!=\sum_{k=1}^{n}(n+1-k)\ln k=(n+1)\ln n!-\sum_{k=1}^{n}k\ln k$$

Simplifying, we get

$$\ln x_n=\frac{2}{n}\sum_{k=1}^{n}k\ln k-\frac{n+1}{n}\ln n!-\frac{n}{2}+\frac{1}{2}\ln n$$

Since

$$A=\lim_{n\to\infty}A_n=\lim_{n\to\infty}\frac{1^12^2\cdots n^n}{n^{n^2/2+n/2+1/12}e^{-n^2/4}}$$

$$\ln x_n=\frac{2}{n}\ln A_n+\left(n+\frac{3}{2}+\frac{1}{6n}\right)\ln n-n-\frac{n+1}{n}\ln n!$$

$$\ln x_n=\frac{2}{n}\ln A_n-\frac{1}{3n}\ln n+1-\frac{n+1}{n}\ln\sqrt{2\pi}-\left(1+\frac{1}{n}\right)O\left(\frac{1}{n}\right)=1-\ln\sqrt{2\pi}$$

Therefore

$$\lim_{n\to\infty}\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{1}{2}}}=\boxed{\frac{e}{\sqrt{2\pi}}}$$

More mathematical constants can be found at: http://www.ebyte.it/library/educards/constants/MathConstants.html

30.2 Famous Inequalities

30.2.1 Cauchy-Schwarz Inequality

Theorem 2 If $f(x)$, $g(x)$ are integrable on $[a,b]$, then:

$$\left(\int_a^b f(x)g(x)dx\right)^2\le\left(\int_a^b f^2(x)dx\right)\cdot\left(\int_a^b g^2(x)dx\right)$$

Proof: Consider

$$\int_a^b[f(x)-kg(x)]^2dx=k^2\int_a^b g^2(x)dx-2k\int_a^b f(x)g(x)dx+\int_a^b f^2(x)dx\ge0$$

Therefore

$$\Delta = 4\left(\int_a^b f(x)g(x)dx\right)^2 - 4\int_a^b f^2(x)dx\int_a^b g^2(x)dx \leq 0$$

Thus

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)dx\right)\cdot\left(\int_a^b g^2(x)dx\right)$$

30.2.2 Young's Inequality

Theorem 3 Let $f(x)$ be continuous and strictly increasing on $[0,+\infty)$ with $f(0)=0, a>0, b=f(a)>0$, then

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy$$

where $f^{-1}(y)$ is the inverse function of $f(x)$. Equality holds if and only if $b=f(a)$.

30.2.3 Holder's Inequality

Theorem 4 If $f(x)$, $g(x)$ are continuous on $[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$ $(p,q>0)$, then

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{p}}$$

30.2.4 Minkowski Inequality

Theorem 5 Let $f(x), g(x) \in R[a,b], 1 \leq p < +\infty$, then

$$\left[\int_a^b \left(|f(x)|+|g(x)|\right)^p dx\right]^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{p}}$$

30.2.5 Hadamard Inequality

Theorem 6 Let $f(x)$ be a concave function on (a,b). For any $x_1, x_2 \in (a,b), x_1 < x_2$, we have

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2} f(t)dt \leq \frac{f(x_1)+f(x_2)}{2}$$

30.2.6 Favard Inequality

Theorem 7 Let $f(x)$ be a non-negative concave function on $[a,b]$. For any $p>1$, we have

$$\frac{1}{b-a}\int_a^b f^p(x)dx \leq \frac{2^p}{p+1}\left(\frac{1}{b-a}\int_a^b f(x)dx\right)^p$$

50 Problems on Limits (Old Version)

Original Author: Zhihu @ 陌亿:https://zhuanlan.zhihu.com/p/464349656

Note: Actually there are 51 problems.

1. $\lim\limits_{x\to 0}\dfrac{\overbrace{\tan\tan\cdots\tan x}^{p\text{ times}}-\overbrace{\sin\sin\cdots\sin x}^{p\text{ times}}}{\tan x-\sin x}$, where $p\in\mathbb{N}^+$

2. $\lim\limits_{x\to+\infty}\left[(a+x)^{1+\frac{1}{x}}-x^{1+\frac{1}{x+a}}\right]$

3. $\lim\limits_{x\to+\infty}\left[\left(x^3+\dfrac{x}{2}-x^3\tan\dfrac{1}{x}\right)e^{\frac{1}{x}}-\sqrt{1+x^6}\right]$

4. $\lim\limits_{x\to\infty}\left(\sqrt[4]{x^4+x^3+x^2+x+1}-\sqrt[3]{x^3+x^2+x+1}\cdot\dfrac{\ln(x+e^x)}{x}\right)$

5. $\lim\limits_{x\to+\infty}\left[x\cdot e^{\frac{1}{x}}\cdot\arctan\dfrac{x^2+x-1}{(x+1)(x+2)}-\dfrac{\pi}{4}\cdot x\right]$

6. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(1+\dfrac{k}{n}\right)\sin\dfrac{k\pi}{n^2}$

7. $\lim\limits_{x\to 0}\dfrac{\sqrt{\frac{1+x}{1-x}}\sqrt[4]{\frac{1+2x}{1-2x}}\cdots\sqrt[2n]{\frac{1+nx}{1-nx}}-1}{3\pi\arctan x-(x^2+1)\arctan^3 x}$

8. $\lim\limits_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}\sqrt[3]{\cos 3x}\cdots\sqrt[n]{\cos nx}}{x^2}$

9. $\lim\limits_{x\to 0^+}\left[\ln(x\ln a)\cdot\ln\left(\dfrac{\ln ax}{\ln\frac{x}{a}}\right)\right], a>1$

10. $\lim\limits_{n\to\infty} n\sin(2\pi n!e)$

11. $\lim\limits_{n\to\infty} n^2\left[n\sin(2e\pi\cdot n!)-2\pi\right]$

12. Let the sequence $\{a_n\}$ satisfy $a_1=1$, and $a_{n+1}=\dfrac{a_n}{(n+1)(a_n+1)}, n\geq 1$. Find the limit $\lim\limits_{n\to\infty} n!a_n$.

13. $\lim\limits_{n\to\infty} \sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}$

14. $\lim\limits_{n\to\infty}\sqrt{n}\left(1-\sum_{k=1}^{n}\dfrac{1}{n+\sqrt{k}}\right)$

15. $A_n=\dfrac{n}{n^2+1}+\dfrac{n}{n^2+2^2}+\cdots+\dfrac{n}{n^2+n^2}$. Find $\lim\limits_{n\to\infty} n\left(\dfrac{\pi}{4}-A\right)$.

16. Let $S_n = \dfrac{\sum_{k=0}^{n} \ln C_n^k}{n^2}$. Find $\lim\limits_{n\to\infty} S_n$.

17. $\lim\limits_{n\to\infty} \prod\limits_{k=1}^{n} \left(1 + \dfrac{k}{n^2}\right)$

18. $\lim\limits_{n\to\infty} \dfrac{1^{an} + 2^{an} + \cdots + n^{an}}{n^{an}}$

19. $\lim\limits_{n\to\infty} \dfrac{n + n^{\frac{1}{2}} + n^{\frac{1}{3}} + \cdots + n^{\frac{1}{n}}}{n}$

20. $\lim\limits_{n\to\infty} \left(\dfrac{1 + 2^p + 3^p + \cdots + n^p}{n^p} - \dfrac{n}{p+1}\right)$, where p is a natural number.

21. $-1 < x_0 < 1, x_n = \sqrt{\dfrac{1 + x_{n-1}}{2}}, n = 1, 2, 3, \cdots$. Find $\lim\limits_{n\to\infty} \prod\limits_{k=1}^{n} x_k$.

22. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{n + 1 - k}{nC_n^k}$

23. $\lim\limits_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2 \left[\dfrac{\pi}{2n}(x_1 + x_2 + \cdots + x_n)\right] dx_1 dx_2 \cdots dx_n$

24. $\lim_{n\to\infty} \sqrt{n} \prod_{k=1}^{n} \frac{e^{1-\frac{1}{k}}}{(1+\frac{1}{k})^k}$

25. $f(x)$ is differentiable at x_0, $a < x_0 < b$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x_0$. Prove that

$$\lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(x_0)$$

26. $\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{k^2}{n^2+k} - \frac{n}{3} \right)$

27. $\lim_{n\to\infty} \frac{n + n^2 + n^3 + \cdots + n^n}{1^n + 2^n + \cdots + n^n}$

28. $\lim_{n\to\infty} \int_0^{\frac{\pi}{3}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$

29. $\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \sin^n x dx$

30. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n-k^2}}$

31. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n^2}\frac{n}{n^2+k^2}$

32. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\sin\frac{2i-1}{n^2}a$

33. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left[\left(n^k+1\right)^{-\frac{1}{k}}+\left(n^k-1\right)^{-\frac{1}{k}}\right]$

34. Let $\alpha_k=\frac{k+\sqrt{k^2+4}}{2}$, find $\lim\limits_{k\to\infty}\prod\limits_{n=1}^{\infty}\left(1-\frac{k}{{\alpha_k}^n+\alpha_k}\right)$

35. $\lim\limits_{n\to\infty}\int_0^1\sum\limits_{k=1}^{n}\frac{\prod_{i=1}^{n}\left(x+\frac{i}{n^2}\right)}{x+\frac{k}{n^2}}dx$

36. $\lim\limits_{n\to\infty}\frac{\ln n}{\ln\left(1^{2020}+2^{2020}+\cdots+n^{2020}\right)}$

37. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n+k^{\alpha}}$

38. $\lim\limits_{n\to\infty}\left(\sin\frac{\ln 2}{2}+\sin\frac{\ln 3}{3}+\cdots+\sin\frac{\ln n}{n}\right)^{\frac{1}{n}}$

39. $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{i+j}{i^2+j^2}$

40. $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left\{ \frac{n}{k} \right\}^2$ where $\{a\}$ denotes the fractional part of a.

41. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n} \right)^k$

42. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n} \right)^{\alpha k}$

43. $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{2^k}{2^{2^k}+1}$

44. $\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots \right)^2$

45. $\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{n} C_n^k \sin^2 \left(\pi\sqrt{k^2+k} \right)$

46. $\lim_{n\to\infty} \frac{\int_0^{\frac{\pi}{2}} \frac{\sin^2(nx)}{\sin x} dx}{\ln n}$

47. $\lim\limits_{x\to\infty}\sum\limits_{k=0}^{n}(-1)^k C_n^k\sqrt{x^2+k}$

48. $\lim\limits_{n\to\infty}\frac{1}{n}\left(\frac{n}{\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}}\right)^n$

49. $\lim\limits_{n\to\infty}\int_0^1\frac{x\sin(nx)}{1+n^6x^2}dx$

50. $\lim\limits_{n\to\infty}\left(\sum\limits_{k=1}^{n}\frac{1}{C_n^k}\right)^n$

51. $\lim\limits_{m\to\infty}\left(\sqrt[m]{m+1}\cdot\sqrt[m^2+m]{\prod\limits_{k=1}^{m}C_m^k}\right)$

50 Limit Problems (New)

Original Author on Zhihu @ 陌亿 :https://zhuanlan.zhihu.com/p/586648267

Note: There are actually 53 problems.

II. Warm-up Problems

1. $a_1 = \sqrt{1+2015}, a_2 = \sqrt{1+2015\sqrt{1+2016}}$,

$\cdots a_n = \sqrt{\left(1+2015\sqrt{1+2016\sqrt{1+\cdots+(2014+n)\sqrt{1+(2013+n)}}}\right)}, \lim_{n\to\infty} a_n$

2. $\lim_{x\to 0^+}\int_0^\infty \frac{e^{-xt}}{\sqrt{t^2+t}}dt$

3. $\lim_{n\to\infty}\frac{\ln n}{n}\left(\frac{\sum_{k=1}^{n-1}\csc\left(\frac{k\pi}{n}\right)}{\ln n}-\frac{2}{\pi}n\right)$

4. $\lim_{n\to\infty} a_n\sum_{k=1}^{n} a_k^2 = 1, \lim_{n\to\infty}\sqrt[3]{n}a_n$

5. If $a_n, b_n > 0$ for all $n \geq 1$, and $\lim_{n\to\infty}\frac{a_{n+1}}{na_n} = a \in \mathbb{R}^+$, $\lim_{n\to\infty}\frac{b_{n+1}}{nb_n} = b \in \mathbb{R}^+$, compute

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{(n!)^3}} \sum_{k=1}^{n} (a_k b_k)^{\frac{1}{k}}$$

6. $\displaystyle \lim_{n\to\infty} n \left[\left(\frac{\sum_{k=1}^{m} a_k^{\frac{1}{n}}}{m} \right)^n - \left(\prod_{k=1}^{n} a_k \right)^{\frac{1}{m}} \right]$

7. $f(x) : [0,1] \to \mathbb{R}$ is an integrable function and continuous at $x = 1$. For $k \geq 1$, find $\displaystyle \lim_{n\to\infty} \frac{1}{n^k} \int_0^1 \left(x + 2^k x^2 + \cdots + n^k x^n \right) f(x)\, dx$

8. $\{b_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers with $b_0 = 1$ and $b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1 + \sqrt{b_{n-1}}}$. Compute $\displaystyle \sum_{n=1}^{\infty} b_n 2^n$

9. $\displaystyle \lim_{n\to\infty} n \left[\left(\frac{1}{\pi} \sum_{k=1}^{n} \sin \frac{\pi}{\sqrt{n^2 + k^2}} \right)^n - \frac{1}{\sqrt[4]{e}} \right]$

10. $f \in \mathbb{R}[0,1]$, find $\displaystyle \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (-1)^k f\left(\frac{k}{n} \right)$

11. $f(x) \in C[0,1]$, $f(x) > 0$, find $\displaystyle \lim_{n\to\infty} \frac{\int_0^1 f^{n+1}(x)\, dx}{\int_0^1 f^n(x)\, dx}$

12. $\lim\limits_{\substack{n\to\infty\\ m\to\infty}} \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{(-1)^{i+j}}{i+j}$

13. $\lim\limits_{n\to\infty} \left\{ \tan\left(\pi\sqrt{n^2+\left\lfloor \frac{6n}{11} \right\rfloor} \right) + 4\sin\left(\pi\sqrt{4n^2+\left\lfloor \frac{8n}{11} \right\rfloor} \right) \right\}$

III. The Real Challenge Begins

1. Let $I_n = \int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} \frac{e^{-{x_{n+1}}^{2n}-{x_{n+2}}^{2n}-\cdots-{x_{2n}}^{2n}} - e^{{x_1}^{2n}-{x_2}^{2n}-\cdots-{x_n}^{2n}}}{{x_1}^{2n}+{x_2}^{2n}+\cdots+{x_n}^{2n}-{x_{n+1}}^{2n}-{x_{n+2}}^{2n}-\cdots-{x_{2n}}^{2n}}\,dx_1dx_2\cdots dx_n$, $\lim\limits_{n\to\infty} I_n$

2. $\lim\limits_{n\to\infty} \left(\sqrt[n+1]{\prod_{k=1}^{n+1}\Gamma\left(\frac{1}{k}\right)} - \sqrt[n]{\prod_{k=1}^{n}\Gamma\left(\frac{1}{k}\right)} \right)$

3. $\lim\limits_{n\to\infty} \left(\max\limits_{0\le x<+\infty} \left| e^{-x} - \frac{1}{\sum_{k=0}^{n}\frac{x^k}{k!}} \right| \right)^{\frac{1}{n}}$

4. $a,b\in\mathbb{R}$, $c\in\mathbb{R}^+$, find $\lim\limits_{n\to\infty}\int_a^b \frac{dx}{c+\prod_{k=0}^{n}\sin^2(x+k)}$

5. $\displaystyle \lim_{n\to\infty}\int_0^{\frac{\pi}{2}} \frac{n}{(\sqrt{2}\cos x)^n+(\sqrt{2}\sin x)^n}\,dx$

6. $\displaystyle \lim_{n\to\infty}\frac{n!}{n^n}\left(\sum_{k=0}^{n}\frac{n^k}{k!}-\sum_{k=n+1}^{\infty}\frac{n^k}{k!}\right)$

7. $f:[0,1]\to\mathbb{R}$ is a continuous function. Find $\displaystyle \lim_{n\to\infty}\int_0^1\cdots\int_0^1 f\left(\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\right)dx_1\cdots dx_n$

8. $a_n=\sqrt{n+a_{n-1}}, a_1=1$. Prove that $\displaystyle a_n=\sqrt{n}+\frac{1}{2}+\frac{1}{8\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)$

9. $\displaystyle \lim_{n\to\infty} n\left(n\int_0^n \frac{\arctan\frac{x}{n}}{x(x^2+1)}\,dx-\frac{\pi}{2}\right)$

10. Let real numbers x,y,z satisfy $e^x+e^y+e^z=2+e^{x+y+z}$. Find

$$\lim_{(x,y,z)\to(0,0,0)}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{x+y+z}{12}\right)$$

11. Let $a_n=\dfrac{\ln n}{n}, m\geq 0, \lambda>0$, determine the constant t such that

$$\lim_{n\to\infty} nt^n\left[\sqrt[n]{\sum_{i=1}^{m}(a_i)^n}-\max_{1\leq k\leq m}\{a_k\}\right]=\lambda$$

12. $\lim\limits_{n\to\infty} n\left(\left(\frac{1}{3\pi}\int_{\pi}^{2\pi}\frac{x}{\arctan(nx)}dx\right)^n - e^{\frac{4}{3\pi^2}}\right)$

13. $\lim\limits_{t\to\infty} t\sum_{k=0}^{\infty}(-1)^k\frac{1}{\sqrt{k^2+t^2}}$

14. $\lim\limits_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 \sin\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)dx_1dx_2\cdots dx_n$

15. Let $A_n=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}$, find

$$\lim_{n\to\infty} n(n(n(n(n(n(-1)^n n!(e(1-A_n)-1)-e)+2e)-5e)+15e)$$

16. $\lim\limits_{n\to\infty} n\left(\left(\int_0^1\frac{1}{1+x^n}dx\right)^n-\frac{1}{2}\right)$

17. $q_n=3^{3^n}+1$, find $\lim\limits_{n\to\infty}\sqrt[3]{6q_0^2+\sqrt[3]{6q_1^2+\sqrt[3]{\cdots\sqrt[3]{6q_n^2}}}}$

18. $\lim\limits_{x\to 0^+}\lim\limits_{y\to+\infty}\frac{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{m=0}^{\infty}\frac{1}{n2^m+1}\int_0^{x^2}\frac{\pi(\sqrt[4]{1+t}-1)\sin t^4}{\sum_{n=1}^{\infty}\frac{((n-1)!)^2(2t)^{2n}}{(2n)!}\int_0^1\frac{(1-2x)\ln(1-x)}{x^2-x+1}dx}}{x^2(x-\tan x)\ln(x^2+1)\left[\left(\frac{2\arctan\frac{x}{x}}{\pi}\right)^y-1\right]}$

19. $\lim\limits_{n\to\infty}\int_0^1\cdots\int_0^1 \dfrac{\left[\sin\left(\frac{\pi}{2}x_1\right)+\sin\left(\frac{\pi}{2}x_2\right)+\cdots+\sin\left(\frac{\pi}{2}x_n\right)\right]^m (x_1^q+\cdots+x_n^q)^s}{\left(\frac{\pi}{2}x_1+\frac{\pi}{2}x_2+\cdots+\frac{\pi}{2}x_n\right)^m (x_1^p+\cdots+x_n^p)^s}dx_1\cdots dx_n$

20. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{\pi k}{2n}\int_0^1 x^{2n}\sin\dfrac{x\pi}{2}dx$

21. Suppose $\lim\limits_{n\to\infty} x_n = +\infty$, the positive series $\sum\limits_{n=1}^{\infty} y_n$ converges, and let n_0 be a natural number. If when $n > n_0$, it holds that $x_n < x_{n+1}, x_n < \dfrac{1}{2}(x_{n-1}+x_{n+1}), y_{n+1} \leq y_n$, prove: $\lim\limits_{n\to\infty}\dfrac{x_n y_n}{x_{n+1}-x_n} = 0$ 22. $\lim\limits_{n\to\infty} n^2\left(\int_0^1 (1+x^n)^{\frac{1}{n}}\,dx - 1\right)$

23. $\lim\limits_{n\to\infty} n\prod\limits_{m=1}^{n}\left(1-\dfrac{1}{m}+\dfrac{5}{4m^2}\right)$

24. For $a, b \in \mathbb{R}^+$, find $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{1}{n+k+b+\sqrt{n^2+kn+a}}$

25. For $k \in \mathbb{N}^*$, find (1) $L = \lim\limits_{n\to\infty}\int_0^1 \dfrac{\ln(1+x^{n+k})}{\ln(1+x^n)}dx$

(2) $\lim\limits_{n\to\infty} n\left(\int_0^1 \dfrac{\ln(1+x^{n+k})}{\ln(1+x^n)}dx - L\right)$

26. For a continuous function $f : [a, b] \to [0, \infty)$, find $\lim\limits_{n\to\infty} n \left(\sqrt[n]{\int_a^b f^{n+1}(x)dx} - \sqrt[n]{\int_a^b f^n(x)dx} \right)$

27. Given $f_0(x) \in R[0, 1]$ and $f_0(x) > 0$, with $f_n(x) = \sqrt{\int_0^x f_{n-1}(t)dt}$ $(n = 1, 2, \cdots)$, find $\lim\limits_{n\to\infty} f_n(x)$

28. For $a, b \in \mathbb{R}$ with $0 < a < b$, $u_0 = a$, $v_0 = b$, $u_{n+1} = \dfrac{u_n + v_n}{2}$, $v_{n+1} = \sqrt{v_{n+1}v_n}$, prove: $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} v_n = \dfrac{b\sin(\arccos(\frac{a}{b}))}{\arccos\frac{a}{b}}$

29. $\lim\limits_{n\to\infty} \dfrac{\pi^n n^n}{\left(2 + \sum_{k=1}^n \frac{4^k(k!)^2}{(2k+1)!}\right)^{2n}}$

30. $\lim\limits_{n\to\infty} \sqrt{n} \left(\int_{-\infty}^{\infty} \dfrac{\cos x}{(1 + x^2)^n} dx \right)$

31. Given $u_1 = 0$, $u_2 = 1$, $u_{n+2} = u_{n+1} + \dfrac{u_n}{2n}$, find $\lim\limits_{n\to\infty} \dfrac{u_n}{\sqrt{n}}$

32. For $a_1 = 1$, $a_n = a_{n-1} + \dfrac{1}{a_{n-1}}$, find $\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{\ln n}(a_n - \sqrt{2n})$

33. Let $f(x), g(x) \in C[a,b]$ with $f(x) \geq 0$, $g(x) > 0$, and $f(x)$ strictly increasing. Prove: (1) For $n \in \mathbb{N}$, when n is sufficiently large, there exists $x_n \in [a,b]$ such that $\dfrac{f^n(x_n)}{\ln n} = n^3 \displaystyle\int_a^b f^n(x)g(x)dx$

(2) $\lim\limits_{n\to\infty} x_n$

34. Given $x_1 > 0$, $x_{n+1} = x_n + \dfrac{n}{\sum_{k=1}^n x_i}$, find $\lim\limits_{n\to\infty} n\left(\sqrt{3} - \dfrac{x_n}{\sqrt{n}}\right)$

35. $\lim\limits_{n\to\infty} \dfrac{\sum_{i=1}^n i^3 (1+\frac{1}{i})^i \cdot \sum_{k=1}^n k^2 \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{k\uparrow 2}}{n^7}$

36. $\lim\limits_{n\to\infty} \dfrac{\ln^2 n}{n} \displaystyle\sum_{k=2}^{n-2} \dfrac{1}{\ln k \cdot \ln(n-k)}$

37. $\lim\limits_{n\to\infty} \dfrac{\sum_{k=1}^n (-1)^k C_n^k \ln k}{\ln(\ln n)}$

38. Let $S_n = 1 + \displaystyle\sum_{k=1}^{n-1} \prod_{i=1}^{k} \dfrac{n-i}{n+i+1}$, find $\lim\limits_{n\to\infty} \dfrac{S_n}{\sqrt{n}}$

39. Let $S_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{2^k(n+k+1)}$ $(n \geq 1)$, find the equivalent infinitesimal of S_n as $n \to \infty$

40. Let $f : [0,1] \to (0, \infty)$ be a function such that $\frac{\ln(f(x))}{x} \in R[0,1]$, and $g : [0,1] \to \mathbb{R}$ be an integrable function continuous at $x = 1$. Prove:

$$\lim_{n\to\infty} n^2 \left(\int_0^1 \sqrt[n]{f(x^n)} g(x) dx - \int_0^1 g(x) dx \right) = g(1) \int_0^1 \frac{\ln f(x)}{x} dx$$

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